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All extremal instantons in Einstein-Maxwell-dilaton-axion theory

Mustapha Azreg-Aïnou*

Başkent University, Department of Mathematics, Bağlıca Campus, Ankara, Turkey

Gérard Clément†

Laboratoire de Physique Théorique LAPTH (CNRS),

B.P.110, F-74941 Annecy-le-Vieux cedex, France

Dmitri V. Gal'tsov‡

Department of Theoretical Physics, Moscow State University, 119899, Moscow, Russia

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We construct explicitly all extremal instanton solutions to $\mathscr{N}=4, D=4$ supergravity truncated to one vector field (Einstein-Maxwell-dilaton-axion (EMDA) theory). These correspond to null geodesics of the target space of the sigma-model $G/H = Sp(4,\mathbb{R})/GL(2,\mathbb{R})$ obtained by compactification of four-dimensional Euclidean EMDA on a circle. They satisfy a no-force condition in terms of the asymptotic charges and part of them (corresponding to nilpotent orbits of the $Sp(4,\mathbb{R})$ U-duality) are presumably supersymmetric. The space of finite action solutions is found to be unexpectedly large and includes, besides the Euclidean versions of known Lorentzian solutions, a number of new asymptotically locally flat (ALF) instantons endowed with electric, magnetic, dilaton and axion charges. We also describe new classes of charged asymptotically locally Euclidean (ALE) instantons as well as some exceptional solutions. Our classification scheme is based on the algebraic classification of matrix generators according to their rank, according to the nature of the charge vectors and according to the number of independent harmonic functions with unequal charges. Besides the nilpotent orbits of G, we find solutions which satisfy the asymptotic no-force condition, but are not supersymmetric. The renormalized on-shell action for instantons is calculated using the method of matched background subtraction.

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*Electronic address: `azreg@baskent.edu.tr`
†Electronic address: `gclement@lapp.in2p3.fr`
‡Electronic address: `galtsov@phys.msu.ru`

I. INTRODUCTION

Gravitational instantons are important ingredients of quantum gravity/supergravity and string theory responsible for their non-perturbative aspects. They may produce a non-trivial topological structure of space-time in the early universe, and play certain role in cosmology. Gravitational instantons exhibiting periodicity in Euclidean time are stationary paths for thermal partition functions, these are responsible for black hole thermodynamics.

Instantons in vacuum Einstein gravity were subject of intense investigations since the late seventies [1, 2], which culminated in their complete topological classification [3]. Instantons in extended supergravities are non-vacuum and typically involve multiplets of scalar and vector fields in four dimensions and form fields in higher dimensions. Non-vacuum gravitational instantons attracted attention in the late eighties, when they were suggested to support the idea of fixing the physical constants via creation of baby universes [4]. Such instantons and Euclidean wormholes were studied within truncated supergravity models containing the axion and the dilaton. In this context, Einstein-axion [5] and Einstein-dilaton-axion [6] wormhole solutions were discovered. These are particular solutions of the Einstein-Maxwell-dilaton-axion theory [7] which is here the subject of more complete investigation. Essentially similar solutions, D-instantons, were then discovered in ten-dimensional IIB supergravity and generalized to any dimensions [8]. Wormhole instanton solutions also exist in four-dimensional Einstein-Maxwell-dilaton theory (without axion) [9]. Non-vacuum gravitational instantons were also explored in the presence of a cosmological constant, in which case they exhibit a de Sitter or anti-de Sitter asymptotic structure [10]. Asymptotically AdS wormholes provide an arena for further study of the ADS/CFT correspondence, which in turn may be used to test the validity of the above proposals of wormhole-induced effects [11]. Apart from various applications directly related to quantum gravity, four-dimensional instantons can be useful in the purely classical theory as a tool to construct new five-dimensional black holes [12].

During the past two decades, instantons including vector fields were extensively studied in $D=4, \mathscr{N}=2$ supergravity. An early work by Whitt [13] and Yuille [14] discussing instantons in Einstein-Maxwell gravity was recently revised and extended by Dunajski and Hartnoll [15] (for a higher-dimensional extension see [16]). These papers deal with the Euclidean counterparts of the Israel-Wilson-Perjès solutions [17]. Euclidean solutions to $\mathscr{N}=2$ supergravity coupled to various matter multiplets were recently studied in detail in a number of papers [18]. General aspects of supersymmetry and dualities in Euclidean supergravities were discussed in [19]. Euclidean supersymmetry and Killing spinor equations in the Euclidean $\mathscr{N}=2$ theory were recently studied with [20] and without a cosmological constant [21]. Instantons in $D=4, \mathscr{N}=1$ supergravity were studied in [22]. In the $\mathscr{N}=4$ case the complete Killing spinor analysis is

available in the Lorentzian sector [23, 24].

The case of Euclidean $D = 4$, $\mathcal{N} = 4$ supergravity in the presence of vector fields was relatively less explored so far, though uncharged axion-dilatonic instantons and wormholes were discovered long ago. In the minimal case this theory contains six vector fields stransforming under $SO(6)$. Here we consider truncation of $\mathcal{N} = 4$ theory to the EMDA theory with only one vector present, leaving the full theory to further work. With this simplification we will be able to give explicitly all extremal instanton solutions, whose variety turns out to be unexpectedly large already in this truncated case. We define four-dimensional extremal instantons as those which have flat three-dimensional slices. Such solutions are characterized by asymptotic charges: the mass M, the NUT parameter N, the electric Q and magnetic P charges, the dilaton charge D and the axion charge A. In the Euclidean theory, the mass, the dilaton charge and the electric charge generate attraction, while the magnetic mass (NUT), the axion and the magnetic charge generate repulsion. Extremality corresponds to fulfilment of the "no force" condition

$$M^2 + D^2 + Q^2 = N^2 + A^2 + P^2, \tag{1.1}$$

which is part of the BPS conditions of $D = 4$, $\mathcal{N} = 4$ supergravity, but does not guarantee supersymmetry, being only the necessary condition for it. We do not investigate the Killing spinor equations here, also leaving this to future work, but we believe that all supersymmetric solutions to the one-vector truncation of $D = 4$, $\mathcal{N} = 4$ supergravity belong to our list.

Here we use purely bosonic tools to identify solutions satisfying the no-force condition (1.1). The method amounts to identify the null geodesic curves of the target space of the sigma model arising upon dimensional reduction of the theory to three dimensions. The underlying heuristic idea relates to the fact that by virtue of the Einstein equations, null-geodesic solutions have Ricci-flat three-metrics which are therefore flat since the Weyl tensor is zero in three dimensions. The method was suggested by one of the present authors in 1986 [25], building on the characterization by Neugebauer and Kramer [26] of solutions depending on a single potential as geodesics of the three-dimensional target space, in the context of five-dimensional Kaluza-Klein (KK) theory. It was further applied in [27] to classify Lorentzian extremal solutions of the EMDA theory and Einstein-Maxwell (EM) theory. In two of these three cases (KK and EMDA) it was found that the matrix generators B of null geodesics split into a nilpotent class ($B^n = 0$ starting with certain integer n), in which cases the matrix is degenerate ($\det B = 0$), and a non-degenerate class ($\det B \neq 0$). The solutions belonging to the first class are regular, while the second class solutions, though still satisfying the no-force constraint (1.1) on asymptotic charges, generically contain singularities. In the EM case all null geodesics are nilpotent orbits, corresponding to the Israel-Wilson-Perjès solutions [27].

This approach partially overlaps (though a bit wider) with the method of nilpotent orbits which was

suggested in [28] (see also [29]) and further developed in [30]. The latter starts with some matrix condition following from supersymmetry, which is generically stronger than our condition selecting the null geodesic subspace of the target space. Our classification includes some a priori non-supersymmetric solutions satisfying the no-force condition (1.1).

The purpose of this paper is to give a complete list of null geodesic instantons of EMDA theory saturating the asymptotic bound (1.1). Technically, we will use the Euclidean version of the EMDA sigma model derived in [31] and equipped with a concise symplectic matrix representation in [32]. In the Lorentzian case, the six-dimensional symmetric target space of the sigma-model obtained by time-like dimensional reduction (appropriate to the generation of black hole solutions) of four-dimensional EMDA is $Sp(4,R)/U(1,1)$, while in the case of space-like reduction it is $Sp(4,R)/U(2)$. In the Euclidean case one finds yet another coset of the same dimensionality six, namely $Sp(4,R)/GL(2,R)$. Our results indicate that the EMDA instanton space is much larger than one could anticipate using analytical continuation of the known Lorentzian solutions. The target space geometry of Euclidean EMDA differs by signature from that of the Lorentzian theory, and contains three independent null directions compared to only two in the Lorentzian case. This gives rise to new classes of three-potential extremal solutions. Also, new classes of solutions arise which are not asymptotically locally flat (ALF) but asymptotically locally Euclidean (ALE). Finally, there are exceptional solutions for which the dilaton diverges at infinity, while the renormalized action is still finite. Generically, the on-shell action for four-dimensional gravitating instantons always diverges at infinity due to slow fall-off of the Gibbons-Hawking term, so it has to be renormalized. As the renormalization tool, we adopt the matched asymptotic subtraction method.

The outline of the paper is as follows. The Euclidean four-dimensional Einstein-three-form-Maxwell-dilaton action is discussed in Sect. II. We perform in Sect. III toroidal reduction of this theory to three dimensions, keeping track of the boundary terms arising in the dualizations involved, which will be relevant for the calculation of the instanton on-shell actions. This reduction leads to a three-dimensional gravitating sigma model with symmetric target space $Sp(4,\mathbb{R})/GL(2,\mathbb{R})$ of signature $(+++---)$. In Sect. IV we construct the matrix representation of this coset and discuss the different asymptotic forms of the coset matrices, leading to ALF and ALE instanton solutions. Some exceptional asymptotics are discovered within each class. In Sect. V we introduce null geodesic solutions. The associated matrix representatives are parametrized in terms of asymptotic charges saturating the Bogomolny bound. We derive a simple formula for the instanton action involving only the boundary values of the scale factor and the dilaton function and their derivatives. We also give a convenient description of the matrix generators in terms of $SO(1,2)$ charge vectors, which will provide the basis for further classification of solutions. Sect. VI presents the classification of ALF instantons which are split into strongly degenerate (nilpotent of rank 2), weakly degenerate

(nilpotent of rank 3) and non-degenerate. The corresponding three kinds of ALE solutions are discussed in Sec. VII. One of these includes a new types of wormhole interpolating between ALE and conical ALF spaces. Examples of ALF and ALE instantons with exceptional asymptotics, including a magnetic linear dilaton solution, are also given. In Sec. VIII we consider the case of multiple independent harmonic functions whose maximal number (three) is determined by the number of independent null directions in target space. The last Sec. IX is devoted to a brief discussion of the six-dimensional uplifting of four-dimensional EMDA instantons. In Appendix A we briefly discuss the sigma-model representation of the "phantom" (with a Maxwell field coupled repulsively to gravity) EMDA model with positive definite signature of the target space. Details of the derivation of solutions with exceptional asymptotics are given in Appendix B, and the proof that multi-potential solutions fall into three distinct classes is given in Appendix C.

II. EUCLIDEAN EMDA THEORY

Recall that the correct choice of the Euclidean action for the axion field in four dimensions follows from positivity requirement and amounts to using initially the three-form field rather than a pseudoscalar axion. The corresponding action with account of the Gibbons-Hawking surface term reads

$$S_0 = \frac{1}{16\pi}\int\limits_{\mathscr{M}} \left(-R\star 1 + 2\mathrm{d}\phi\wedge\star\mathrm{d}\phi + 2\mathrm{e}^{-4\phi}H\wedge\star H + 2\mathrm{e}^{-2\phi}F\wedge\star F\right) - \frac{1}{8\pi}\int\limits_{\partial\mathscr{M}} \mathrm{e}^{\psi/2}K\star\mathrm{d}\Phi\,, \tag{2.1}$$

where $F = \mathrm{d}A$ is the Maxwell two-form and H is the three-form field strength related to the two-form potential B via the relation involving the Chern-Simons term:

$$H = \mathrm{d}B - A\wedge F. \tag{2.2}$$

The four-dimensional Hodge dual is denoted by a star $\star$ and defined in local coordinates by

$$\star(\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_m}) = \frac{1}{(4-m)!}E^{\mu_1\cdots\mu_m}{}_{\mu_{m+1}\cdots\mu_4}\mathrm{d}x^{\mu_{m+1}}\wedge\cdots\wedge\mathrm{d}x^{\mu_4}\,,\; m\le 4\,, \tag{2.3}$$

where the totally anti-symmetric symbol $\varepsilon_{\mu\nu\rho\sigma}$ and tensor $E_{\mu\nu\rho\sigma}$ are related by $E_{\mu\nu\rho\sigma} = g^{1/2}\varepsilon_{\mu\nu\rho\sigma}$, $E^{\mu\nu\rho\sigma} = g^{-1/2}\varepsilon^{\mu\nu\rho\sigma}$, with $\varepsilon_{1234} = \varepsilon^{1234} = +1$. The boundary $\partial\mathscr{M}$, which is embedded in $\mathscr{M}$, is described by $\Phi(x^\mu)\equiv 0$, while $\mathrm{e}^{\psi/2}$ is a scale factor ensuring that $\mathrm{e}^{\psi/2}\mathrm{d}\Phi$ measures the proper distance in a direction normal to $\partial\mathscr{M}$, and K is the trace of the extrinsic curvature of $\partial\mathscr{M}$ in $\mathscr{M}$.

The corresponding field equations are

$$\mathrm{d}\star\left[\mathrm{e}^{-4\phi}\left(\mathrm{d}B - A\wedge F\right)\right] = 0, \tag{2.4}$$

$$\mathrm{d}\star\left(\mathrm{e}^{-2\phi}F\right) = 0, \tag{2.5}$$

$$\Box\phi = \frac{1}{4}\,\mathrm{e}^{-2\phi}F^2 + \frac{1}{6}\,\mathrm{e}^{-4\phi}H^2. \tag{2.6}$$

It follows that $A=0$ is a consistent truncation of EMDA which is pure dilaton-axion gravity, while it is not consistent to set $B=0$ and $\phi=0$ (constraints on F will be produced), so Einstein-Maxwell theory is not a consistent truncation of EMDA. Of course this does not preclude the possibility of EMDA solutions with zero dilaton and axion fields, which satisfy the arising constraints on the vector field.

To pass to the pseudoscalar axion consistently, one has to ensure the Bianchi identity for H:

$$\mathrm{dd}B = \mathrm{d}(H + A\wedge F) = 0\,. \tag{2.7}$$

This is achieved adding to the action (2.1) a new term with the Lagrange multiplier κ

$$S_\kappa = \frac{1}{8\pi}\int_{\mathscr{M}'} \kappa\, \mathrm{d}(H + A\wedge F) = \frac{1}{8\pi}\int_{\mathscr{M}'} \kappa\,(\mathrm{d}H + F\wedge F)\,, \tag{2.8}$$

where $\mathscr{M}'$ is $\mathscr{M}$ with the monopole sources of H (where the Bianchi identity (2.7) breaks down) cut out. In what follows, we will use both the open set $\mathscr{M}'$ and the original manifold $\mathscr{M}$ as the integration domains, keeping in mind that in absence of magnetic singularities integrals of both types will be the same. Transforming the first term as

$$\int_{\mathscr{M}'} \kappa \mathrm{d}H = \int_{\mathscr{M}'} \mathrm{d}\,(\kappa H) - \int_{\mathscr{M}'} \mathrm{d}\kappa\wedge H, \tag{2.9}$$

we obtain a boundary term from the total derivative, while the H-dependent part of the bulk action will be

$$S_H = \frac{1}{8\pi}\int \left(\mathrm{e}^{-4\phi} H\wedge\star H - \mathrm{d}\kappa\wedge H\right). \tag{2.10}$$

Treating H as fundamental field rather than the B-field strength, we obtain varying S_H over H:

$$H = \frac{1}{2}\mathrm{e}^{4\phi}\star\mathrm{d}\kappa. \tag{2.11}$$

Eliminating H in favor of κ then gives

$$S_H = -\frac{1}{32\pi}\int \mathrm{e}^{4\phi}\mathrm{d}\kappa\wedge\star\mathrm{d}\kappa. \tag{2.12}$$

Thus the sum $S_0 + S_\kappa$ gives the Euclidean bulk action in terms of the pseudoscalar axion

$$S_E = \frac{1}{16\pi}\int\limits_{\mathscr{M}} \left(-R\star 1 + 2\mathrm{d}\phi\wedge\star\mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{4\phi}\mathrm{d}\kappa\wedge\star\mathrm{d}\kappa + 2\mathrm{e}^{-2\phi}F\wedge\star F + 2\kappa F\wedge F\right), \tag{2.13}$$

or in the component form

$$S_E = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{g}\left[-R + 2\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mathrm{e}^{4\phi}\partial_\mu\kappa\partial^\mu\kappa + \mathrm{e}^{-2\phi}F_{\mu\nu}F^{\mu\nu} - \kappa F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{2.14}$$

where $\tilde{F}^{\mu\nu} = -E^{\mu\nu\rho\sigma}F_{\rho\sigma}/2$ (the unusual minus sign is due to our convention $\varepsilon_{1234} = +1$, which is usually taken as $\varepsilon_{0123} = +1$). The last term in (2.14) coupling the axion to the pseudoscalar Maxwell invariant does not depend on the metric, so it does not contribute to the Einstein equations:

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\,\mathrm{e}^{4\phi}\partial_\mu\kappa\partial_\nu\kappa + \mathrm{e}^{-2\phi}(2F_{\lambda\mu}F^{\lambda}{}_{\nu} - \frac{1}{2}F_{\lambda\tau}F^{\lambda\tau}g_{\mu\nu})\,. \tag{2.15}$$

Our purpose will be to solve the equations of motion for extremal instantons and to calculate the on-shell action for them. However, the latter is usually divergent. The boundary manifold $\partial\mathscr{M}$ will generically consist of an external boundary $\partial\mathscr{M}_r$ corresponding to some finite value of a suitably chosen radial coordinate r, and an inner boundary $\partial\mathscr{M}_{\mathrm{int}}$ which must be introduced if solution under consideration has curvature singularities. In this paper we will consider ALF and/or ALE solutions and send the external boundary $\partial\mathscr{M}_r$ to infinity ($r\to\infty$) at the end of the calculation. It is well-known that already in absence of the inner boundary, such a calculation is ambiguous since in four space-time dimensions the gravitational contribution generically diverges in the limit $r\to\infty$. The matter part of the action can also diverge (see Sects. IV and VII). This problem was encountered in many applications related to gravitational instantons and black hole thermodynamics which corresponds to black hole solutions periodic in imaginary time.

Two main tools were suggested to overcome this difficulty. One simple way to renormalize the action consists in subtracting the (also infinite) action calculated for a reference solution (background) which must be matched with the solution in question on the boundary. This leads to a finite action, but the result will depend on the particular chosen background. Moreover, it is not always possible to embed the boundary geometry in the background space-time exactly, so one has to resort to an approximate embedding. The matching procedure then becomes non-trivial, as revealed the discussion of the late nineties [33–36] of the action for the Taub-NUT and Taub-bolt instantons given earlier by Gibbons and Perry [2]. While direct omission of the divergent term in the action led to the values $S_{TN} = 4\pi N^2$ and $S_{Tb} = 5\pi N^2$ for the self-dual Taub-NUT instanton and the Taub-bolt [2], the choice of self-dual Taub-NUT as background leads respectively to $S_{TN} = 0$ and $S_{Tb} = \pi N^2$ [33].

Both of the above values for the Taub-NUT instanton action can be given physical meaning [37, 38] within the second regularization scheme for gravity-coupled theories which was first suggested [39, 40], in the context of AdS/CFT correspondence, and later developed into the holographic renormalization technique [41, 42]. This method consists in adding to the D-dimensional bulk action a $(D-1)$-dimensional counterterm action, depending uniquely on the boundary geometry, which cancels the divergences of the on-shell gravitational action, background subtraction being then unnecessary [37, 39]. This proposal was first formulated in the case of AdS asymptotics [39], where the corresponding boundary stress-tensor has an interpretation as the vacuum expectation value of the stress-tensor operator of the quantum field theory

holographically dual to the bulk gravity-coupled theory. It has since been realized that a similar procedure remains valid in the asymptotically flat limit of an infinite curvature radius of the AdS, and different proposals for counterterm actions were made [37, 38, 43, 44], which all cancel divergences. Moreover, the asymptotically flat case may also admit a holographic interpretation [45, 46] in the context of M-theory. The advantage of the method of geometric counterterms (apart from its direct relation to holography) is that it removes divergences without subtraction, and is thus independent of the choice of a reference solution. This method ascribes to the vacuum self-dual Taub-NUT instanton the non-zero Gibbons-Perry value of the action $S_{TN} = 4\pi N^2$ [38, 40, 43, 44]. Accordingly, the holographically renormalized action for the Taub-bolt instanton also corresponds to the "naive" subtraction of the Minkowskian value of the divergent term. However, the drawback of the holographic renormalization of the action in the case of ALF and/or ALE Euclidean instantons is that the general procedure suggested for generating counterterms is not unique, and the finite part remaining after cancellation of divergences is therefore also ambiguous [37, 38, 43, 44, 47]. Another drawback is that the result may depend on the choice of coordinates [37, 38].

In this paper we evaluate in a first step the actions for different extremal instantons, without discussing possible applications to quantum gravity/holography. For this purpose, we shall adopt the conceptually and technically simpler procedure of Hunter [33] which prescribes subtraction of a suitably matched vacuum background solution, which will be self-dual Taub-NUT in the case of ALF solutions. It will therefore give by definition the value zero for the vacuum Taub-NUT instantons. Other instantons endowed with electric, magnetic and scalar charges will lead to a non-zero action, these may be regarded as excitations of vacuum Taub-NUT, similarly to the Taub-bolt instanton in the pure gravity case.

Since the bulk term in (2.13) is zero for the reference space-time, to renormalize the action it will be enough to subtract the matched background integrals over $\partial\mathscr{M}_r$ in the Gibbons-Hawking term and the potentially divergent axion boundary term replacing K by $[K] = K - K_0$, and $\kappa e^{4\phi} \star d\kappa$ by $[\kappa e^{4\phi} \star d\kappa] = \kappa e^{4\phi} \star d\kappa - \kappa_0 e^{4\phi_0} \star d\kappa_0$, where K_0 is the reference value of the trace of extrinsic curvature evaluated for an appropriate background metric g_0 solving the field equations and matched to the considered metric on the boundary $\partial\mathscr{M}_r$, and κ_0, ϕ_0 are analogous matter background terms. Collecting then all the boundary terms, we obtain

$$ {}^4S_b = \frac{1}{16\pi}\int\limits_{\partial\mathscr{M}'} [\kappa e^{4\phi} \star d\kappa] - \frac{1}{8\pi}\int\limits_{\partial\mathscr{M}} e^{\psi/2}[K] \star d\Phi, \tag{2.16} $$

where the pull-back of the three-form $\star d\kappa$ onto the boundary $\partial\mathscr{M}$ is understood. Note that the bulk matter action in the form (2.14) is not positive definite in contrast to (2.1): the difference is hidden in the boundary term.

Some of our new instantons contain inner singularities, for which the above infrared renormalization

is not sufficient and the integrals still diverge in the UV. In such cases the instanton action will remain undetermined.

III. KALUZA-KLEIN REDUCTION WITH BOUNDARY TERMS

To develop generating technique for instantons we apply dimensional reduction to three dimensions. The derivation of the EMDA sigma model was first given in [31] and further developed in [32]. Its Euclidean version can be obtained by analytic continuation, but we want in addition to keep boundary terms which arise upon dualization of the Kaluza-Klein one-form and the three-dimensional Maxwell one-form and which were disregarded in earlier work. We therefore present here a more complete derivation.

Consider an oriented manifold $\mathscr{M}$ with a Riemannian metric $g_{\mu\nu}$ admitting an $U(1)$ isometry generated by the Killing vector field $\xi = \partial_t$ where t now is an Euclidean coordinate with period β. The metric reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f(dt - \omega_i dx^i)^2 + \frac{1}{f} h_{ij} dx^i dx^j \,, \tag{3.1}$$

where f, ω_i, h_{ij} are functions of x^i $(i = 1, 2, 3)$. Occasionally we will also use an exponential parametrization of the scale factor $f = e^{-\psi}$. The classification of such metrics in terms of the fixed point sets of ξ was given by Gibbons and Hawking [1]. In four-dimensional space, the submanifolds on which the norm $\xi^\mu \xi_\mu = f(x^i)$ vanishes may be two-dimensional (bolts), or zero-dimensional (nuts). A regular foliation of space in terms of the orbits of ξ is possible on the manifold $\mathscr{M}'$ obtained from $\mathscr{M}$ after subtracting the sets of its fixed points. The contribution to the action of the nuts and bolts can be computed in a general way using either Kaluza-Klein reduction [33], or ADM decomposition with an associated Hamiltonian formalism [34]. Our Lagrangian is not of the form considered in [34], so the results of this paper can not be directly applied here. Rather we would need for this purpose the Euclidean version of the Hamiltonian formulation of EMDA presented in [48]. We postpone Hamiltonian analysis for a future paper, and here restrict ourselves to Kaluza-Klein reduction to three dimensions, with careful account for the boundary terms.

The dimensional reduction of the gravitational action is standard. Since the t coordinate parametrizes the circle of circumference β, the integral over $\mathscr{M}$ reduces to the integral over a three-dimensional Euclidean space $\mathscr{E}$ with two-dimensional boundary $\partial\mathscr{E}$ described by $\Psi(x^i) = \Phi(x^\mu)|_{\partial\mathscr{E}} \equiv 0$. Throughout this paper the three-dimensional Hodge dual is denoted by an asterisk $*$ and defined by a similar formula to (2.3) where we replace 4 by 3 and $\sqrt{g}$ by $\sqrt{h}$ and set the conventions $\varepsilon_{123} = \varepsilon^{123} = +1$. Reducing R à la Kaluza-Klein we obtain

$$-\frac{1}{16\pi}\int\limits_{\mathscr{M}} R \star 1 = \frac{\beta}{16\pi}\int\limits_{\mathscr{E}} \left(-\mathscr{R} * 1 + \frac{1}{2} * d\psi \wedge d\psi + \frac{1}{2} e^{-2\psi} * \mathscr{F} \wedge \mathscr{F} \right) + {}^3S_{b1} \,, \tag{3.2}$$

where $\omega = \omega_i \mathrm{d}x^i$ and $\mathscr{F} = \mathrm{d}\omega$ are the Kaluza-Klein one- and two-form, respectively. The boundary term due to the total derivative reads

$$ {}^3S_{b1} = \frac{\beta}{16\pi}\int\limits_{\partial\mathscr{E}} *\mathrm{d}\psi\,. \tag{3.3} $$

On the other hand, dimensional reduction of the regularized four-dimensional Gibbons-Hawking term leads to

$$ {}^4S_{GH} = -\frac{1}{8\pi}\int\limits_{\partial\mathscr{M}} \mathrm{e}^{\psi/2}[K]\star\mathrm{d}\Phi = -\frac{\beta}{16\pi}\int\limits_{\partial\mathscr{E}} \left(2[k]*\mathrm{d}\Psi + *[\mathrm{d}\psi]\right), \tag{3.4} $$

where $[k]$ is the regularized trace of the extrinsic curvature of $\partial\mathscr{E}$ embedded in $\mathscr{E}$. The last term partly cancels the contribution from the reduction of the bulk gravity (3.3), however there is a residual gravitational contribution to the two-dimensional boundary term

$$ {}^3S_{bG} = \frac{\beta}{16\pi}\int\limits_{\partial\mathscr{E}} \left(-2[k]*\mathrm{d}\Psi + *\mathrm{d}\psi_0\right). \tag{3.5} $$

In the matter part of the action (2.13) we have to keep in mind the transgression rules. Equipping temporarily the four-dimensional Maxwell potential and the two-form strength with a hat, $\mathrm{d}\hat{F} = \hat{A}$, and denoting the corresponding three-dimensional forms as A, F, we define (to simplify notation we will temporary use electric and magnetic potentials differing from those in the previous section by $\sqrt{2}$, they will be rescaled back at the end of the calculation)

$$ \hat{A} = v\mathrm{d}t + A, \qquad F = \mathrm{d}v\wedge\omega + \mathrm{d}A\,, \tag{3.6} $$

so that

$$ \hat{F} = \mathrm{d}v\wedge\vartheta + F, \qquad \vartheta = \mathrm{d}t - \omega. \tag{3.7} $$

Then the four-dimensional and the three-dimensional duals will be related via

$$ \star\hat{F} = \mathrm{e}^{\psi} * \mathrm{d}v + \mathrm{e}^{-\psi} * F\wedge\vartheta\,. \tag{3.8} $$

Substituting this into the Eq. (2.13) and combining with (3.2) we obtain the bulk action

$$ \begin{aligned} S_E = \frac{\beta}{16\pi}\int\limits_{\mathscr{E}} \Big[& -\mathscr{R}*1 + \frac{1}{2}*\mathrm{d}\psi\wedge\mathrm{d}\psi + \frac{1}{2}\mathrm{e}^{-2\psi}*\mathscr{F}\wedge\mathscr{F} + 2*\mathrm{d}\phi\wedge\mathrm{d}\phi \\ & -\frac{1}{2}\mathrm{e}^{4\phi}*\mathrm{d}\kappa\wedge\mathrm{d}\kappa + 2\mathrm{e}^{-2\phi}\left(*\mathrm{d}v\wedge\mathrm{d}v\mathrm{e}^{\psi} + *F\wedge F\mathrm{e}^{-\psi}\right) + 4\kappa\mathrm{d}v\wedge F\Big]\,. \end{aligned} \tag{3.9} $$

Now we wish to dualize the three-dimensional Maxwell two-form

$$ F = *g, \tag{3.10} $$

where g is the one form $g = g_i \mathrm{d}x^i$. To guarantee the validity of the Bianchi identity $\mathrm{d}(F - \mathrm{d}v \wedge \omega) = 0$ we add to the action (3.9) a new term with Lagrange multiplier u (the magnetic potential)

$$S_u = \frac{\beta}{4\pi} \int\limits_{\mathscr{E}'} u\, \mathrm{d}(F - \mathrm{d}v \wedge \omega) = \frac{\beta}{4\pi} \int\limits_{\mathscr{E}'} [-\mathrm{d}u \wedge F + u \mathrm{d}v \wedge \mathscr{F}] + \frac{\beta}{4\pi} \int\limits_{\partial \mathscr{E}'} uF \,. \tag{3.11}$$

Now consider the last two terms in the action (3.9) together with the $\mathrm{d}u \wedge F$ term in the volume part of (3.11), and perform variation over g as a fundamental field. We obtain

$$g = (\mathrm{d}u - \kappa \mathrm{d}v)\, \mathrm{e}^{2\phi + \psi} \,. \tag{3.12}$$

This can be now substituted back into the action (3.9) to eliminate the Maxwell three-dimensional two-form in favor of the scalar magnetic potential u

$$S_E = \frac{\beta}{16\pi} \int\limits_{\mathscr{E}} \Big[-\mathscr{R} * 1 + \frac{1}{2} * \mathrm{d}\psi \wedge \mathrm{d}\psi + \frac{1}{2} \mathrm{e}^{-2\psi} * \mathscr{F} \wedge \mathscr{F} + 2 * \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} \mathrm{e}^{4\phi} * \mathrm{d}\kappa \wedge \mathrm{d}\kappa$$
$$+ 2\mathrm{e}^{\psi - 2\phi} * \mathrm{d}v \wedge \mathrm{d}v - 2\mathrm{e}^{\psi + 2\phi} * (\mathrm{d}u - \kappa \mathrm{d}v) \wedge (\mathrm{d}u - \kappa \mathrm{d}v) + 4u \mathrm{d}v \wedge \mathscr{F} \Big] \,. \tag{3.13}$$

The last dualization is that of the Kaluza-Klein two-form. For this we write $\mathscr{F} = *\varpi$ and add a new term, with Lagrange multiplier η, ensuring the Bianchi identity $\mathrm{d}\mathscr{F} = 0$:

$$S_\eta = \frac{\beta}{16\pi} \int\limits_{\mathscr{E}'} \eta \mathrm{d}\mathscr{F} = \frac{\beta}{16\pi} \int\limits_{\mathscr{E}'} (-\mathrm{d}\eta \wedge \mathscr{F}) + \frac{\beta}{16\pi} \int\limits_{\partial \mathscr{E}'} \eta \mathscr{F} \,. \tag{3.14}$$

Collecting all $\mathscr{F}$-terms in (3.13) and performing variation over ϖ as a fundamental field, we obtain

$$\varpi = \mathrm{e}^{2\psi} (\mathrm{d}\eta - 4u \mathrm{d}v) = \mathrm{e}^{2\psi} (\mathrm{d}\chi - 2u \mathrm{d}v + 2v \mathrm{d}u) \,, \quad (\chi \equiv \eta - 2uv) \,. \tag{3.15}$$

Substituting this back in (3.13) we eliminate $\mathscr{F}$ from the action replacing it with the NUT potential χ.

To make contact with the notation of [31] we rescale the electric and magnetic potentials $u \to u/\sqrt{2}$, $v \to v/\sqrt{2}$. The defining equations (3.6), (3.12) and (3.15) for v, u, χ become, in component notation

$$F_{i4} = \frac{1}{\sqrt{2}} \partial_i v \,, \tag{3.16}$$

$$\mathrm{e}^{-2\phi} F^{ij} - \kappa \tilde{F}^{ij} = \frac{f}{\sqrt{2h}} \varepsilon^{ijk} \partial_k u \,, \tag{3.17}$$

$$\partial_i \chi + v \partial_i u - u \partial_i v = -f^2 h_{ij} \frac{\varepsilon^{jkl}}{\sqrt{h}} \partial_k \omega_l \tag{3.18}$$

(here again the unusual minus sign in front of κ is due to our convention $\varepsilon_{1234} = +1$). The full bulk action is that of the gravity-coupled three-dimensional sigma model

$$S_\sigma = -\frac{\beta}{16\pi} \int \mathrm{d}^3 x \sqrt{h} \left(\mathscr{R} - G_{AB} \partial_i X^A \partial_j X^B h^{ij} \right) \,, \tag{3.19}$$

where the target space variables are $\mathbf{X} = (f, \phi, v, \chi, \kappa, u)$, integration is over the three-space $\mathscr{E}$ and the target space metric $\mathrm{d}l^2 = G_{AB}\mathrm{d}X^A\mathrm{d}X^B$ reads

$$\mathrm{d}l^2 = \frac{1}{2}\, f^{-2}\mathrm{d}f^2 - \frac{1}{2}\, f^{-2}(\mathrm{d}\chi + v\mathrm{d}u - u\mathrm{d}v)^2 + f^{-1}\mathrm{e}^{-2\phi}\mathrm{d}v^2 - f^{-1}\mathrm{e}^{2\phi}(\mathrm{d}u - \kappa\mathrm{d}v)^2 + 2\mathrm{d}\phi^2 - \frac{1}{2}\,\mathrm{e}^{4\phi}\mathrm{d}\kappa^2 . \quad (3.20)$$

This space has the isometry group $G = Sp(4,\mathbb{R})$, the same as its Lorentzian counterpart [31], in which case

$$\mathrm{d}l_L^2 = \frac{1}{2}\, f^{-2}\mathrm{d}f^2 + \frac{1}{2}\, f^{-2}(\mathrm{d}\chi + v\mathrm{d}u - u\mathrm{d}v)^2 - f^{-1}\mathrm{e}^{-2\phi}\mathrm{d}v^2 - f^{-1}\mathrm{e}^{2\phi}(\mathrm{d}u - \kappa\mathrm{d}v)^2 + 2\mathrm{d}\phi^2 + \frac{1}{2}\,\mathrm{e}^{4\phi}\mathrm{d}\kappa^2 . \quad (3.21)$$

The Euclidean line element (3.20) is derived from (3.21) by the following complexification:

$$v \to \mathrm{i}v, \quad \chi \to \mathrm{i}\chi, \quad \kappa \to -\mathrm{i}\kappa. \quad (3.22)$$

The metric (3.20) is the metric on the coset G/H, whose nature can be uncovered from a signature argument. The Killing metric of $sp(4,\mathbb{R}) \sim so(3,2)$ algebra has the signature $(+6,-4)$, with plus standing for non-compact and minus for compact generators. Since the signature of the target space is $(+3,-3)$, it is clear that the isotropy subalgebra contains three non-compact and one compact generators. Such a subalgebra of $so(3,2) \sim sp(4,\mathbb{R})$ is $\mathrm{lie}\,(H) \sim so(2,1) \times so(1,1) \sim gl(2,\mathbb{R})$. We therefore deal with the coset $SO(3,2)/(SO(2,1)\times SO(1,1)) = Sp(4,\mathbb{R})/GL(2,\mathbb{R})$.

In addition to the bulk action we have a number of surface terms resulting from three-dimensional dualizations as well as from dimensional reduction of the four-dimensional Gibbons-Hawking-axion term. Collecting these together, and taking care of the rescaling of the electric and magnetic potentials, we get:

$$S_{\mathrm{inst}} = {}^3S_b = \frac{\beta}{16\pi}\int\limits_{\partial\mathscr{E}} (-2[k]*\mathrm{d}\Psi + *\mathrm{d}\psi_0) + \frac{\beta}{16\pi}\int\limits_{\partial\mathscr{E}'} \Big([\kappa\mathrm{e}^{4\phi} * \mathrm{d}\kappa] + 2\sqrt{2}uF + (\chi + uv)\mathscr{F}\Big) . \quad (3.23)$$

Note that the *on-shell* value of the action which we are interested in for instantons is entirely given by the boundary term ${}^3S_b$ since the bulk sigma-model action vanishes by virtue of the contracted three-dimensional Einstein equations

$$\mathscr{R}_{ij} = G_{AB}\partial_i X^A \partial_j X^B . \quad (3.24)$$

Variation of the bulk action (3.19) over X^A gives the equations of motion

$$\partial_i\left(\sqrt{h}h^{ij}G_{AB}\partial_j X^B\right) = \frac{1}{2}\, G_{BC,A}\partial_i X^B \partial_j X^C h^{ij}\sqrt{h}, \quad (3.25)$$

which can be rewritten in a form explicitly covariant both with respect to the three-space metric h_{ij}, and to the target space metric G_{AB}

$$\nabla_i J^i_A = 0\,, \quad (3.26)$$

where ∇_i is the total covariant derivative involving Christoffel symbols both of h_{ij} and G_{AB}. The six currents associated with the potentials read

$$J^i_A = h^{ij}\partial_j X^B G_{AB}\,. \tag{3.27}$$

Note that, according to (3.20), the NUT potential χ is a cyclic target space coordinate, so the corresponding current satisfies the conservation equation with an odinary derivative

$$\partial_i\left(\sqrt{h}h^{ij}J_{j\chi}\right) = 0\,, \quad J_{i\chi} = G_{\chi A}\partial_i X^A\,, \tag{3.28}$$

and defines a conserved quantity, the NUT charge

$$N = -\int\limits_{\partial\mathscr{E}} \sqrt{\sigma} n^i J_{i\chi} \mathrm{d}^2x = \frac{1}{2}\int\limits_{\partial\mathscr{E}} \sqrt{\sigma} f^{-2} n^i(\partial_i\chi + v\partial_i u - u\partial_i v)\mathrm{d}^2x. \tag{3.29}$$

where $\partial\mathscr{E}$ is any topological two-sphere and n^i is the outward normal.

IV. MATRIX REPRESENTATION

To proceed, we have to introduce the matrix representation of the coset $Sp(4,\mathbb{R})/GL(2,\mathbb{R})$. The symplectic group $Sp(4,\mathbb{R})$ is the group of real 4×4 matrices M satisfying

$$M^T JM = J\,, \quad J = \begin{bmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{bmatrix}\,, \tag{4.1}$$

where σ_0 is the 2×2 identity matrix. The group $Sp(4,\mathbb{R})$ has three maximal subgroups [49] of dimension four, the compact subgroup $U(2)$, and the two non-compact subgroups $U(1,1)$ and $GL(2,\mathbb{R})$, leading to the three cosets associated with three-dimensional reductions of EMDA: $Sp(4,\mathbb{R})/U(2) = SO(3,2)/(SO(3)\times SO(2))$ (reduction of Lorentzian EMDA relative to a spacelike Killing vector, or of phantom Lorentzian EMDA relative to a timelike Killing vector [50]), $Sp(4,\mathbb{R})/U(1,1) = SO(3,2)/(SO(2,1)\times SO(2))$ (reduction of Lorentzian EMDA relative to a timelike Killing vector), and $Sp(4,\mathbb{R})/GL(2,\mathbb{R}) = SO(3,2)/(SO(2,1)\times SO(1,1))$ (reduction of normal or phantom Euclidean EMDA).

We will use the representation [32] for the $sp(4,\mathbb{R})$ algebra

$$V_a = \frac{1}{2}\begin{pmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{pmatrix}\,, \quad W_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}\,, \tag{4.2}$$

$$U_a = \frac{1}{2}\begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}\,, \quad U_2 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\,, \tag{4.3}$$

($a = 0,1,3$, $\sigma_1 = \sigma_x$, $\sigma_2 = i\sigma_y$, $\sigma_3 = \sigma_z$ with σ_x, σ_y, σ_z the Pauli matrices). The matrices V_a, W_a are symmetric and the four matrices U_a, U_2 are antisymmetric. An extensive discussion of the internal algebraic structures in this matrix space can be found in [27].

The isotropy subgroup $H = GL(2,R)$ for Euclidean EMDA leaves invariant a given fixed point $\mathbf{X}$ of the target space. It is convenient to choose this point to be the point at infinity $\mathbf{X}(\infty)$, which will depend on the boundary conditions. We shall assume that the three-space $\mathscr{E}$ is asymptotically flat and topologically $\mathbb{R}^3$, so that the asymptotic three-metric is, in spherical coordinates,

$$\mathrm{d}\sigma^2 \equiv h_{ij}\mathrm{d}x^i\mathrm{d}x^j \simeq \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\,. \tag{4.4}$$

The possible asymptotic behaviors for $r = |\mathbf{r}| \to \infty$ of the three-dimensional fields $\mathbf{X}(\mathbf{r})$ can in principle be derived from the analysis of the three-dimensional field equations or, equivalently, by a discussion of the possible algebraic types of the matrix representing a given point of target space. In the generic ALF case, the asymptotic four-dimensional metric is

$$\mathrm{d}s^2 \simeq (\mathrm{d}t - 2N\cos\theta\,\mathrm{d}\varphi)^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\,, \tag{4.5}$$

and $f(\infty) = 1$, while the five other target space coordinates go to zero

$$\mathbf{X}(\infty) = (1,0,0,0,0,0)\,. \tag{4.6}$$

We will start with this asymptotic behavior as basis to build our matrix representation of $Sp(4,\mathbb{R})/GL(2,\mathbb{R})$, then discuss the non-ALF cases, which are connected to the ALF case by group transformations.

In our representation of $sp(4,\mathbb{R})$ the generators (4.2) are non-compact, while (4.3) are compact. We can choose the $GL(2,\mathbb{R})$ subalgebra to be spanned by

$$\mathrm{lie}\,(H) = (V_a, U_2)\,, \tag{4.7}$$

while (U_a, W_a) will be the generators of the coset. The infinitesimal transformations generated by (4.7) leave invariant the real 4×4 matrix

$$\eta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}\,. \tag{4.8}$$

A symmetric matrix representative M of the coset such that $M(\infty) = \eta$ for the ALF asymptotic behavior (4.6) has the block structure

$$M = \begin{pmatrix} P^{-1} & P^{-1}Q \\ QP^{-1} & -P + QP^{-1}Q \end{pmatrix}\,, \tag{4.9}$$

with the same Q as in [32], but a new P, namely

$$P = \mathrm{e}^{-2\phi}\begin{pmatrix} f\mathrm{e}^{2\phi}+v^2 & v \\ v & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} vw-\chi & w \\ w & -\kappa \end{pmatrix} \quad (w = u-\kappa v). \tag{4.10}$$

The 2×2 block matrices in (4.9) are given by

$$P^{-1} = \begin{pmatrix} f^{-1} & -f^{-1}v \\ -f^{-1}v & \mathrm{e}^{2\phi}+f^{-1}v^2 \end{pmatrix}, \quad P^{-1}Q = \begin{pmatrix} -f^{-1}\chi & f^{-1}u \\ f^{-1}v\chi + w\mathrm{e}^{2\phi} & -\kappa\mathrm{e}^{2\phi} - f^{-1}vu \end{pmatrix},$$

$$-P+QP^{-1}Q = \begin{pmatrix} -f-v^2\mathrm{e}^{-2\phi}+w^2\mathrm{e}^{2\phi}+f^{-1}\chi^2 & -v\mathrm{e}^{-2\phi}-\kappa w\mathrm{e}^{2\phi}-f^{-1}u\chi \\ -v\mathrm{e}^{-2\phi}-\kappa w\mathrm{e}^{2\phi}-f^{-1}u\chi & -\mathrm{e}^{-2\phi}+\kappa^2\mathrm{e}^{2\phi}+f^{-1}u^2 \end{pmatrix}. \tag{4.11}$$

We note that the matrix (4.9) is not symplectic, but antisymplectic:

$$M^T J M = -J\,. \tag{4.12}$$

However this is enough to ensure that the matrix current

$$J^i = h^{ij} M^{-1}\partial_j M \tag{4.13}$$

is symplectic. The matrix (4.9) can be obtained from the corresponding matrix in [31] as follows. Analytical continuation (3.22), together with multiplication by i of the second row and column of the 2×2 blocks P and Q leads to the blocks P and iQ. Then multiplication by $-$i of the second row and column of the matrix M in [31] leads to (4.9). In terms of M the target space metric (3.20) will read

$$\mathrm{d}l^2 = -\frac{1}{4}\,\mathrm{tr}\left(\mathrm{d}M\mathrm{d}M^{-1}\right) = \frac{1}{2}\,\mathrm{tr}\left[(P^{-1}\mathrm{d}P)^2 - (P^{-1}\mathrm{d}Q)^2\right], \tag{4.14}$$

while the sigma-model field equations (3.26) read

$$\boldsymbol{\nabla}\left(M^{-1}\boldsymbol{\nabla}M\right) = 0\,, \tag{4.15}$$

where $\boldsymbol{\nabla}$ stands for the three–dimensional covariant derivative, and the scalar product with respect to the metric h_{ij} is understood.

If the ALF restriction (4.6) is raised, the representative matrix (4.9) will go at infinity to an arbitrary constant symmetric antisymplectic matrix

$$M(\infty) = A \tag{4.16}$$

different from η. Generically this matrix will be of the form (4.9) where the fields f, ϕ, etc. are replaced by their (arbitrary) values at infinity $f(\infty)$, $\phi(\infty)$, etc. As the scalar potentials ψ, ϕ, v, as well as the

pseudoscalar potentials χ, κ, u, are only defined up to an additive constant, the generic $M(\infty)$ can always be gauge-transformed to the ALF form η. An exceptional $M(\infty)$, which is not gauge-equivalent to η, is one for which (4.9) breaks down (at infinity) because $P^{-1}(\infty)$ is not invertible, i.e. $\det(P^{-1})(\infty) \equiv (f^{-1}\mathrm{e}^{2\phi})(\infty) = 0$. This can be subclassified according to the rank of $P^{-1}(\infty)$. Rank 1 corresponds to either $f^{-1}(\infty) = 0$ or $\mathrm{e}^{2\phi}(\infty) = 0$, while rank 0 ($P^{-1}(\infty) = 0$) corresponds to both vanishing. Let us discuss briefly these three possible exceptional asymptotic behaviors (more details are given in Appendix B):

Case E1 (ALE). In the case $f^{-1}(\infty) = 0$, the asymptotic solution of the sigma model field equations (4.15), which is

$$M(r) \simeq A(I + Br^{-1}) \tag{4.17}$$

(with B a constant symplectic matrix) leads to $f^{-1} = M_{11} \simeq O(r^{-1})$, which may be normalized to

$$f \simeq r\,. \tag{4.18}$$

As shown in Appendix B, the asymptotic coset representative A can be gauge transformed to $A = \eta_1'$ with

$$\eta_1' = \begin{pmatrix} 0 & 0 & \pm1 & 0 \\ 0 & 1 & 0 & 0 \\ \pm1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.19}$$

It follows that $\chi = -M_{13} f$ goes asymptotically to $\chi \simeq \mp f$ which, with asymptotically vanishing electromagnetic potentials v and u, is dualized using (3.18) to $\omega_\varphi \simeq \pm\cos\theta$. The resulting asymptotic four-dimensional metric (3.1) is recognized to be the four-dimensional Euclidean metric in three-spherical coordinates

$$ds^2 \simeq \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_3^2 = \mathrm{d}\rho^2 + \frac{\rho^2}{4}[\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 + (\mathrm{d}\eta \mp \cos\theta \mathrm{d}\varphi)^2]\,, \tag{4.20}$$

with the angular coordinate $\eta = t$, and the radial coordinate $\rho = (4r)^{1/2}$. This is the ALE case.

Case E2. In the case $\mathrm{e}^{2\phi}(\infty) = 0$ with $f^{-1}(\infty) \neq 0$, the asymptotic matrix representative can be gauge transformed to

$$\eta_2' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp1 \\ 0 & 0 & -1 & 0 \\ 0 & \mp1 & 0 & 0 \end{pmatrix}. \tag{4.21}$$

This is an exceptional ALF case.

Case E3. In the case $P^{-1}(\infty) = 0$, the asymptotic matrix representative can be gauge transformed to the block form

$$\eta_3' = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}, \quad \beta^2 = 1, \tag{4.22}$$

as shown in Appendix B. This includes an exceptional ALE subcase E3a, with

$$\beta_{\rm a} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \tag{4.23}$$

and a one-parameter subcase E3b, with

$$\beta_{\rm b} = \begin{pmatrix} \cos\nu & \sin\nu \\ \sin\nu & -\cos\nu \end{pmatrix}, \tag{4.24}$$

interpolating between a second ALE behavior for $\sin\nu = 0$, and a magnetic linear dilaton asymptotic behavior with linearly rising gravitational, dilaton and magnetic potentials (the magnetic Euclidean equivalent of the electric linear dilaton behavior in Lorentzian EMDA [48]) for $\cos\nu = 0$.

V. NULL GEODESIC SOLUTIONS

In the following we will use the formalism developed in [27, 32]. For the reader's convenience we reproduce here basic results. Starting with the sigma-model action in the matrix form

$$S_\sigma = -\frac{\beta}{16\pi}\int d^3x\sqrt{h}\left\{\mathscr{R} + \frac{1}{4}\,{\rm tr}\,(\boldsymbol{\nabla} M\boldsymbol{\nabla} M^{-1})\right\}, \tag{5.1}$$

we obtain the equations of motion (4.15) together with the three–dimensional Einstein equations

$$\mathscr{R}_{ij} = -\frac{1}{4}\,{\rm tr}\,\left(\nabla_i M\nabla_j M^{-1}\right). \tag{5.2}$$

As was noticed by Neugebauer and Kramer [26], when one makes the special assumption that all target space coordinates X^A depend on x^i through only one scalar potential, i.e. $X^A = X^A[\tau(x^i)]$, it follows from the equation of motion that this potential can be chosen to be harmonic[1],

$$\Delta\tau = 0, \qquad \Delta = \boldsymbol{\nabla}^2, \tag{5.3}$$

Eq. (3.25) reducing then to the geodesic equation on the target space

$$\frac{d^2X^A}{d\tau^2} + \Gamma^A_{BC}\frac{dX^B}{d\tau}\frac{dX^C}{d\tau} = 0. \tag{5.4}$$

[1] Note that if τ is harmonic in three dimensions ($\Delta_3\tau = 0$), it will be harmonic in four dimensions ($\Delta_4\tau = 0$) as well, since $\Delta_4 = \sqrt{g}^{-1}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu = f\sqrt{h}^{-1}\partial_i\sqrt{h}h^{ij}\partial_j = f\Delta_3$.

This may be rewritten in matrix terms as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(M^{-1}\frac{\mathrm{d}M}{\mathrm{d}\tau}\right)=0\,, \tag{5.5}$$

and first integrated by

$$M^{-1}\frac{\mathrm{d}M}{\mathrm{d}\tau}=B\,, \tag{5.6}$$

where $B\in\mathrm{lie}\,(G)\ominus\mathrm{lie}\,(H)$ is a constant matrix. A second integration leads to the solution to the geodesic equation in the exponential form

$$M=A\,\mathrm{e}^{B\tau}\,, \tag{5.7}$$

with $A\in G/H$ another constant matrix. The parametrisation (5.7) reduces the three–dimensional Einstein equations (5.2) to

$$\mathscr{R}_{ij}=\frac{1}{4}\,(\mathrm{tr}\,B^2)\nabla_i\tau\nabla_j\tau\,. \tag{5.8}$$

From this expression it is clear that in the particular case

$$\mathrm{tr}\,B^2=0 \tag{5.9}$$

the three–space is Ricci–flat. In three dimensions the Riemann tensor is then also zero, and consequently the three–space $\mathscr{E}$ is flat. We shall assume in the following that $\mathscr{E}=\mathbb{R}^3$. From Eq. (4.14) one can see that the condition (5.9) corresponds to null geodesics [25] of the target space

$$\mathrm{d}l^2=\frac{1}{4}\,(\mathrm{tr}\,B^2)\,\mathrm{d}\tau^2=0\,. \tag{5.10}$$

An important feature of the target space of Euclidean EMDA (3.20) as compared to that of the Lorentzian theory (3.21) is that it has now the signature $(+,+,+,-,-,-)$ with three, rather than two, independent null directions. Each null direction gives rise to some BPS solution which is potentially supersymmetric within a suitable supergravity embedding. One new minus sign is associated with the twist potential, reflecting the possibility of extremal Taub-NUT solutions (and consequently multi-Taub-NUTs). Another new minus sign is related to the axion field, reflecting the possibility of extremal instantons without Maxwell fields. At the same time, the electric direction has now a positive definite metric component, while the magnetic one remains negative. So, in absence of twist and axion field, only magnetic or dyonic configurations can be extremal.

Our boundary conditions imply that the harmonic potential $\tau(x^i)$ goes to a constant value at infinity, which we can take to be zero by a redefinition of the matrix A. Then, these solutions are null target space geodesics going through the point $A=M(\infty)$. In the following we discuss the ALF case, with $A=\eta$,

$$M=\eta\,\mathrm{e}^{B\tau}\,. \tag{5.11}$$

Null geodesic going through other points $A = \eta'$ corresponding to exceptional asymptotics,

$$M' = \eta' \, \mathrm{e}^{B'\tau} \,, \tag{5.12}$$

can be generated from (5.11) by $Sp(4,\mathbb{R})$ transformations

$$M' = K^T M K \,, \quad B' = K^{-1} B K \,. \tag{5.13}$$

In the ALF case, the generators of the coset are W_a, U_a, so that one can write

$$B = 2(\alpha^a W_a + \beta^a U_a) \,, \tag{5.14}$$

where α^a, β^a are constants depending on the charges. These charges are the mass M and NUT charge N, the dilaton and axion charges D, A and the electric and magnetic charges Q, P defined, as in the Lorentzian case [27], by the following behavior of the target space variables at spatial infinity:

$$\begin{aligned} f \sim 1 - \frac{2M}{r} \,, &\qquad \chi \sim -\frac{2N}{r} \,, \\ \phi \sim \frac{D}{r} \,, &\qquad \kappa \sim \frac{2A}{r} \,, \\ v \sim \frac{\sqrt{2}Q}{r} \,, &\qquad u \sim \frac{\sqrt{2}P}{r} \,. \end{aligned} \tag{5.15}$$

Using the representation (4.2), (4.3) for the generators we obtain B in the following block form:

$$B = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \,, \quad a = \alpha^a \sigma_a \,, \; b = \beta^a \sigma_a \; (a = 0,1,3) \,, \tag{5.16}$$

with symmetric 2×2 blocks a, b. Assuming that the monopole harmonic function is normalized to $\tau = r^{-1}$, and comparing with (5.15), we can express the coefficients and the matrices in (5.16) in terms of the charges:

$$\alpha^a = (M + D, \; -\sqrt{2}Q, \; M - D) \,, \quad \beta^a = (N - A, \; \sqrt{2}P, \; N + A) \,, \tag{5.17}$$

$$a = \begin{pmatrix} 2M & -\sqrt{2}Q \\ -\sqrt{2}Q & 2D \end{pmatrix} \,, \quad b = \begin{pmatrix} 2N & \sqrt{2}P \\ \sqrt{2}P & -2A \end{pmatrix} \,. \tag{5.18}$$

Note that the dualized one-forms $\mathrm{d}\kappa$, g and ϖ may be extracted from the lower left-hand block of $M^{-1}\mathrm{d}M = B\mathrm{d}\tau$:

$$(M^{-1}\mathrm{d}M)_{21} = -P^{-1}\mathrm{d}QP^{-1} = \begin{pmatrix} \varpi & -g - v\varpi \\ -g - v\varpi & \mathrm{e}^{4\phi}\mathrm{d}\kappa + 2vg + v^2\varpi \end{pmatrix} = -b\,\mathrm{d}\tau \,. \tag{5.19}$$

In particular, $\varpi = -2N\mathrm{d}\tau$, leading after inverse dualization to

$$\omega = *\varpi = 2N \cos\theta \mathrm{d}\varphi \tag{5.20}$$

for all monopole ALF geodesic solutions.

The matrix B is identically traceless:

$$\operatorname{tr} B \equiv 0\,, \tag{5.21}$$

while for B^2 one obtains

$$\operatorname{tr} B^2 = 4\left[(\alpha^a)^2 - (\beta^a)^2\right], \tag{5.22}$$

where the Euclidean norm is understood: $(\alpha^a)^2 = (\alpha^0)^2 + (\alpha^1)^2 + (\alpha^3)^2$. Therefore the null geodesic condition $\operatorname{tr} B^2 = 0$ translates into $(\alpha^a)^2 = (\beta^a)^2$, or in terms of charges

$$M^2 + D^2 + Q^2 = N^2 + A^2 + P^2\,. \tag{5.23}$$

This no-force condition for instantons can be obtained from that in the Lorentzian sector [27]

$$M^2 + N^2 + D^2 + A^2 = Q^2 + P^2\,. \tag{5.24}$$

by the complexification

$$Q \to \mathrm{i}Q\,, \quad N \to \mathrm{i}N\,, \quad A \to -\mathrm{i}A\,. \tag{5.25}$$

corresponding to (3.22).

In the space of charges the group $H = SO(2,1)\times SO(1,1)$ is operating as a duality symmetry, so it is convenient to replace the Euclidean vectors α^a, β^a by the $SO(2,1)$ vectors

$$\boldsymbol{\mu} = (\mu^0, \overrightarrow{\mu}) \equiv (\beta^0, \overrightarrow{\alpha})\,, \quad \boldsymbol{\nu} = (\nu^0, \overrightarrow{\nu}) \equiv (\alpha^0, \overrightarrow{\beta})\,, \tag{5.26}$$

with $\overrightarrow{\alpha} \equiv (\alpha^1, \alpha^3)$ (similarly for other variables). In terms of the charges,

$$\boldsymbol{\mu} = (N - A,\ -\sqrt{2}Q,\ M - D), \quad \boldsymbol{\nu} = (M + D,\ \sqrt{2}P,\ N + A)\,. \tag{5.27}$$

With this new parametrization, (5.14) takes the form

$$B(\boldsymbol{\mu}, \boldsymbol{\nu}) = 2(\mu^0 U_0 + \mu^1 W_1 + \mu^3 W_3 + \nu^0 W_0 + \nu^1 U_1 + \nu^3 U_3)\,. \tag{5.28}$$

The condition (5.23) now reads

$$\boldsymbol{\mu}^2 = \boldsymbol{\nu}^2\,, \tag{5.29}$$

with the $SO(2,1)$ norm:

$$\boldsymbol{\mu}^2 = \eta^{ab}\mu_a\mu_b\,, \quad \eta^{ab} = \operatorname{diag}(-1,1,1)\,. \tag{5.30}$$

This leads to

$$B^2 = 2\begin{pmatrix} \lambda^1\sigma_3 - \lambda^3\sigma_1 & \lambda^0\sigma_2 \\ \lambda^0\sigma_2 & \lambda^1\sigma_3 - \lambda^3\sigma_1 \end{pmatrix}, \tag{5.31}$$

where $\boldsymbol{\lambda}$ is the skew product:

$$\boldsymbol{\lambda} = \boldsymbol{\mu}\wedge\boldsymbol{\nu}, \quad \lambda_a = \varepsilon_{abc}\mu^b\nu^c, \quad \varepsilon_{013} = +1. \tag{5.32}$$

For the matrix B^3 one finds

$$B^3 = 2B[\boldsymbol{\nu}\wedge\boldsymbol{\lambda}, \boldsymbol{\mu}\wedge\boldsymbol{\lambda}], \tag{5.33}$$

leading to

$$\mathrm{tr}\, B^3 = 0. \tag{5.34}$$

In view of (5.21), (5.9) and (5.34), the characteristic equation for B reduces to

$$B^4 + (\det B)I = 0, \tag{5.35}$$

so that B^4 is proportional to the unit 4×4 matrix:

$$B^4 = 4\boldsymbol{\lambda}^2 I. \tag{5.36}$$

In terms of the charges, using (5.29),

$$\boldsymbol{\lambda}^2 = [\boldsymbol{\mu}\cdot\boldsymbol{\nu} - \boldsymbol{\mu}^2][\boldsymbol{\mu}\cdot\boldsymbol{\nu} + \boldsymbol{\mu}^2] = -[2m_+d_- - q_-^2][2m_-d_+ - q_+^2], \tag{5.37}$$

with

$$m_\pm = M \pm N, \quad d_\pm = D \pm A, \quad q_\pm = Q \pm P. \tag{5.38}$$

Note that the algebraic properties (5.21), (5.9), (5.34) and (5.35) of the matrix B, which have been established in the ALF case, are also valid in the case of exceptional asymptotic behaviors, the corresponding B matrices being related to those of the ALF case by the similarity transformations (5.13).

Finally we evaluate the boundary action (3.23) for null-geodesic solutions. This is the sum $S_{\text{inst}} = S_1 + S_2$ of two surface integrals. The first, purely gravitational contribution (3.5), is the sum of the regularized Gibbons-Hawking term (3.4) and the boundary integral (3.3) for the background solution, both evaluated on a large sphere of radius R. For an ALF metric of the form (3.1) with $\omega_i \mathrm{d}x^i = -2N\cos\theta\,\mathrm{d}\varphi$, the appropriate background [33] is self-dual Taub-NUT

$$\mathrm{d}s_0^2 = f_0(r_0)(\mathrm{d}t_0 - 2N_0\cos\theta\,\mathrm{d}\varphi)^2 + f_0^{-1}(r_0)[\mathrm{d}r_0^2 + r_0^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)], \tag{5.39}$$

with $f_0(r_0) = (1+2|N_0|/r_0)^{-1}$, and the matching conditions are $t_0 = m(R)t$, $r_0 = m^{-1}(R)r$, $N_0 = m(R)N$, with $m(R) = (f(R)/f_0(m^{-1}R))^{1/2} = 1 + (M-|N|)/R + O(R^{-2})$. The regularized trace of the extrinsic curvature of $\partial\mathscr{M}$ is, in the monopole case,

$$[K] = K - K_0 = f^{1/2}(r)\left(k(r) - \frac{1}{2}f^{-1}(r)f'(r)\right) - f_0^{1/2}(r_0)\left(k_0(r_0) - \frac{1}{2}f_0^{-1}(r_0)f'(r_0)\right), \tag{5.40}$$

with the extrinsic curvatures of $\partial\mathscr{E}$ for the solution and the background

$$k(r) = \frac{2}{r}, \quad k_0(r_0) = \frac{2}{r_0} = \left(\frac{f(R)}{f_0(m^{-1}R)}\right)^{1/2}\frac{2}{r}. \tag{5.41}$$

The net regularized extrinsic curvature $[k]$ of $\partial\mathscr{E}$ ($r=R$) is thus zero, so that (3.5) reduces to

$$S_1 = -\frac{\beta}{16\pi}\lim_{R\to\infty}\int_{r=R}\sqrt{h}\mathrm{d}\sigma\, f^{-1/2}f_0^{-1/2}f_0' = -\frac{\beta|N|}{2}. \tag{5.42}$$

The second surface integral is that of the contributions of the various dualizations evaluated on the boundary $\partial\mathscr{E}'$, which has two disjoint components, a large sphere at infinity with the normal oriented outwards, and a small sphere shielding the source $r=0$ of the harmonic potential $\tau = 1/r$ (we will later generalize to the case of multi-center harmonic potentials) with the normal oriented inwards:

$$\begin{aligned} S_2 &= \frac{\beta}{16\pi}\oint_{\partial\mathscr{E}'}\sqrt{h}\mathrm{d}\sigma[\mathrm{e}^{4\phi}\kappa\kappa_r' + f^{-2}(\chi+uv)(\chi_r' + vu_r' - uv_r') + 2f^{-1}\mathrm{e}^{2\phi}u(u_r' - \kappa v_r')] \\ &= \frac{\beta}{4}\left[\mathrm{e}^{4\phi}\kappa\dot{\kappa} + f^{-2}(\chi+uv)(\dot{\chi} + v\dot{u} - u\dot{v}) + 2f^{-1}\mathrm{e}^{2\phi}u(\dot{u} - \kappa\dot{v})\right]_{\tau=0}^{\tau=\infty}, \end{aligned} \tag{5.43}$$

where $\dot{}$ is the derivation relative to τ. This may be evaluated using the first integral (5.6). The upper left-hand corner block of (5.6) gives

$$-\dot{P}P^{-1} + QP^{-1}\dot{Q}P^{-1} = B_{11} \equiv a. \tag{5.44}$$

Tracing the different terms yields

$$-\mathrm{tr}\,(\dot{P}P^{-1}) = \frac{\dot{\Delta}}{\Delta} \qquad (\Delta \equiv \det(P^{-1}) = f^{-1}\mathrm{e}^{2\phi}), \tag{5.45}$$

$$\mathrm{tr}\,(QP^{-1}\dot{Q}P^{-1}) = \mathrm{e}^{4\phi}\kappa\dot{\kappa} + f^{-2}(\chi+uv)(\dot{\chi} + v\dot{u} - u\dot{v}) + 2f^{-1}\mathrm{e}^{2\phi}u(\dot{u} - \kappa\dot{v}). \tag{5.46}$$

This last term is the integrand of (5.43). So,

$$S_2 = \frac{\beta}{4}\left[\mathrm{tr}\,(a) - \frac{\dot{\Delta}}{\Delta}\right]_{\tau=0}^{\tau=\infty} = -\frac{\beta}{4}\left[\frac{\dot{\Delta}}{\Delta}\right]_{\tau=0}^{\tau=\infty}. \tag{5.47}$$

Summing (5.42) and (5.47) leads to the total action

$$S_{\rm inst} = \frac{\beta}{4}\left(-\left[\frac{\dot{\Delta}}{\Delta}\right]_{\tau=0}^{\tau=\infty} - 2|N|\right). \tag{5.48}$$

This shall be evaluated later in the various cases.

VI. DISCUSSION OF THE SOLUTIONS: ALF ASYMPTOTICS

The generic matrix B satisfying Eq. (5.35) is regular (non-degenerate) and of rank 4. If B is singular (degenerate), $\det B = 0$, the exponential in (5.11) reduces to a polynomial of third degree if $\mathrm{rank}\, B = 3$ (we will qualify this case as "weakly degenerate") or, as we shall see, of first order if $\mathrm{rank}\, B = 2$, which will be the "strongly degenerate" case. In this section we will investigate these three classes of null geodesic solutions in the case of ALF asymptotics, and treat the case of exceptional asymptotics in the next section.

A. Strongly degenerate case

According to (5.36) B is degenerate if $\boldsymbol{\lambda}$ is lightlike,

$$\boldsymbol{\lambda}^2 = 0\,. \tag{6.1}$$

On account of (5.29), this corresponds to the condition on the charge vectors,

$$(\boldsymbol{\mu} + \varepsilon \boldsymbol{\nu})^2 = 0\,, \quad \varepsilon = \pm 1\,, \tag{6.2}$$

or, in terms of the charges

$$2m_\varepsilon d_{-\varepsilon} - q_{-\varepsilon}^2 = 0\,. \tag{6.3}$$

Contrary to the Lorentzian case [27], the vanishing (6.1) of the square of the $SO(2,1)$ vector $\boldsymbol{\lambda}$ does not imply the vanishing of this vector itself. Thus, Eq (5.29) along with the generic condition (6.2) lead to the weakly degenerate case $\mathrm{rank}\, B = 3$. However, if the stronger condition

$$\boldsymbol{\lambda} = 0 \tag{6.4}$$

is satisfied along with (5.29), then $\mathrm{rank}\, B = 2$ and $B^2 = 0$. In this strongly degenerate case, the matrix M depends linearly on τ

$$M = \eta\,(I + B\tau)\,. \tag{6.5}$$

Comparing with (4.9), one obtains

$$\begin{aligned}
&f = (1 + 2M\tau)^{-1}\,, \quad \chi = -2Nf\tau = \frac{N}{M}(f-1)\,,\\
&e^{2\phi} = 1 + 2D\tau - 2Q^2 f\tau^2\,, \quad \kappa = 2e^{-2\phi}\tau(A - PQf\tau)\,,\\
&v = \sqrt{2}Qf\tau\,, \quad u = \sqrt{2}Pf\tau\,.
\end{aligned} \tag{6.6}$$

In the special case of a one-center harmonic function $\tau = 1/r$ the resulting metric is, on account of (5.20),

$$\mathrm{d}s^2 = \left(1+\frac{2M}{r}\right)^{-1}(\mathrm{d}t - 2N\cos\theta\,\mathrm{d}\varphi)^2 + \left(1+\frac{2M}{r}\right)\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right]. \tag{6.7}$$

This generically has non-zero Ricci tensor, with scalar curvature and Kretchmann invariant

$$R = g^{\mu\nu}R_{\mu\nu} = \frac{2(M^2-N^2)}{r(2M+r)^3}\,, \tag{6.8}$$

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{44(M^2-N^2)^2 + 64M(M^2-N^2)r + 48(M^2+N^2)r^2}{r^2(2M+r)^6}\,, \tag{6.9}$$

so that there is a curvature singularity at $r=0$, unless $N = \pm M$ (see below), in which case (6.7) is a regular vacuum metric, namely (anti-)self-dual Taub-NUT. We also note that the result (6.6) implies

$$\Delta = 1 + 2(M+D)\tau + 2(2MD - Q^2)\tau^2\,, \tag{6.10}$$

so that, depending on the values of the charges, this linear solution may also develop a singularity for a finite value of τ. Excluding curvature singularities for a finite r by the constraint $M > 0$, we find that (6.10) inserted in (5.48) always leads to a finite action

$$S_{\mathrm{inst}} = \frac{\beta}{2}(M - |N| + D) \tag{6.11}$$

for a single center, and

$$S_{\mathrm{inst}} = \frac{n\beta}{2}(M - |N| + D) \tag{6.12}$$

for a multi-center solution $\tau = \sum_{i=1}^{n} 1/|\mathbf{r} - \mathbf{r}_i|$, irrespective of the possible presence of singularities of $\mathrm{e}^{2\phi}$. In the vacuum case $D = A = Q = P = 0$ and $N = \pm M$ from the no-force condition, so that the one-center solution reduces to the self-dual Taub-NUT instanton with vanishing action.

The strong degeneracy condition $\boldsymbol{\lambda} = 0$ holds if the two vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are collinear, with either one of the vectors vanishing as limiting cases. The generic condition

$$\boldsymbol{\nu} = c\boldsymbol{\mu}\,, \quad c = -P/Q\,, \tag{6.13}$$

splits into two subcases:

1) If the vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are not necessarily lightlike, one must have $c = -\varepsilon$ in view of (6.2). This implies

$$N = -\varepsilon M\,, \quad A = \varepsilon D\,, \quad P = \varepsilon Q\,, \tag{6.14}$$

so that only three of the charges are independent. These solutions, where the no-force condition (5.23) is solved by independently balancing each electric-type charge by an equal magnetic-type charge, generalize

the Taub-NUT instantons of [1]. In the case of a one-center harmonic function $\tau = 1/r$, the corresponding metric is the vacuum (anti-) self-dual Taub-NUT

$$\mathrm{d}s^2 = \left(1 + \frac{2M}{r}\right)^{-1} (\mathrm{d}t + 2\varepsilon M\cos\theta \mathrm{d}\varphi)^2 + \left(1 + \frac{2M}{r}\right) \left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\right] , \tag{6.15}$$

where to remove the Misner string singularity one must identify t with the period $8\pi M$. More generally, both the Maxwell field and the axidilaton fields are separately self-dual, so that the corresponding energy-momentum tensors vanish. The relations (6.6) and (6.14) lead for the dilaton-axion system to

$$\kappa = \varepsilon(1 - \mathrm{e}^{-2\phi}) , \tag{6.16}$$

implying cancellation of the scalar terms at the right hand side of the four-dimensional Einstein equations (2.15), and for the Maxwell system to

$$\tilde{F}_{i4} = \frac{1}{\sqrt{2}} \mathrm{e}^{2\phi} (\kappa\partial_i v - \partial_i u) = -\frac{\varepsilon}{\sqrt{2}} \partial_i v = -\varepsilon F_{i4} , \tag{6.17}$$

leading to cancellation of the Maxwell terms. Therefore the subcase 1 strongly degenerate solution represents the self-dual EMDA dressing of the Ricci-flat self-dual Taub-NUT instanton, with the finite action

$$S_{\mathrm{inst}} = 4\pi|N|D \tag{6.18}$$

(except in the case of a cylindrical spacetime, $|N| = M = 0$, in which case $S_{\mathrm{inst}} = \beta D/2$ with β arbitrary). To our knowledge, this non-vacuum instanton has not appeared in the literature before (its Lorentzian counterpart, however, is known [27, 51]).

Actually, this subcase should be divided into three sectors, according to the sign of the pseudonorm

$$\boldsymbol{\mu}^2 = \boldsymbol{\nu}^2 = 2(Q^2 - 2MD) . \tag{6.19}$$

1a) Timelike sector ($Q^2 < 2MD$). All solutions of this sector can be generated by $SO(2,1)$ transformations from the neutral $\vec{\mu} = \vec{\nu} = 0$ solution with $P = Q = 0$, $A = -N = \varepsilon D = \varepsilon M$. This sector can be further divided into future and past (for the vector $\boldsymbol{\nu}$). In the future timelike sector ($M > 0$, $D > 0$), (6.6) shows that the exponentiated dilaton $\mathrm{e}^{2\phi}$ and the metric function f are obviously positive for all positive τ, so that these solutions are regular for a multicenter harmonic function

$$\tau = \sum_{i=1}^{s} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} , \tag{6.20}$$

with equal residues to ensure absence of Misner strings if t is periodically identified with period $8\pi M$. This is the EMDA dressed generalisation of the multi-Taub-NUT instanton of Gibbons and Hawking. In the past timelike sector ($M < 0$, $D < 0$), both $\mathrm{e}^{2\phi}$ and f develop a singularity for a finite positive value of τ.

1b) Lightlike sector ($Q^2 = 2MD$). This relation is reminiscent of a similar relation in the Lorentzian sector [27] $d = -q^2/2m$, with the complex charges $q = Q + \mathrm{i}P$, $m = M + \mathrm{i}N$, $d = D + \mathrm{i}A$. So in some sense the solutions of this sector can be considered as analytic continuations of stationary extremal solutions to EMDA. Again, this sector can be divided in a future lightlike sector (M and D positive), with regular multi-Taub-NUT instantons as above, and a past lightlike sector (M and D negative) where $\mathrm{e}^{2\phi}$ and f become singular for a finite positive value of τ.

1c) Spacelike sector ($Q^2 > 2MD$). All the solutions of this sector, which can be generated by $SO(2,1)$ transformations from the neutral solution with $P = Q = 0$, $A = N = \varepsilon D = -\varepsilon M$, lead to a singular $\mathrm{e}^{2\phi}$.

2) If the vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are lightlike ($\boldsymbol{\mu}^2 = \boldsymbol{\nu}^2 = 0$), then $c \neq \pm 1$ remains an arbitrary parameter. In addition to (5.23) one has two more constraints on the charges

$$(D+M)Q = (A-N)P\,, \quad (D-M)P = (A+N)Q\,, \tag{6.21}$$

so that again only three of the charges are independent (note that the relations (6.14) also solve the conditions (6.21)). Other relations between the charges (which follow from the preceding) are

$$D^2 - A^2 = M^2 - N^2 = (P^2 - Q^2)/2\,, \quad PQ = AM - ND\,. \tag{6.22}$$

This subcase includes in the limits $c \to 0$ and $c \to \infty$:

— A 2-charge family of purely electric solutions if $\boldsymbol{\nu} = 0$, with

$$N^2 = M^2 + Q^2/2\,, \quad D = -M\,, \quad A = -N\,, \quad P = 0\,. \tag{6.23}$$

These solutions have negative action

$$S_{\mathrm{inst}} = -4\pi N^2 \tag{6.24}$$

(with $\beta = 8\pi|N|$), which can be correlated with the fact that, from Eq. (6.10),

$$\mathrm{e}^{2\phi} = f[1 - 4N^2\tau^2]\,, \tag{6.25}$$

showing that they develop a singularity for a finite value of τ.

— A 2-charge family of purely magnetic solutions if $\boldsymbol{\mu} = 0$, with

$$N^2 = M^2 - P^2/2\,, \quad D = M\,, \quad A = N\,, \quad Q = 0\,. \tag{6.26}$$

These solutions are regular for $f > 0$ (but singular for $f = 0$ if $P \neq 0$), with

$$\mathrm{e}^{-2\phi} = f\,, \quad v = 0\,, \quad \chi = \kappa = -\sqrt{2}\frac{N}{P}u = \frac{N}{M}(f-1)\,. \tag{6.27}$$

A special class in this subcase is that of neutral solutions with $P = Q = 0$. Then, the relations $\boldsymbol{\mu}^2 = \boldsymbol{\nu}^2 = 0$ and $\boldsymbol{\nu} = c\boldsymbol{\mu}$ are solved by

$$N = \varepsilon' M\,, \quad A = \varepsilon' D\,, \quad P = Q = 0\,, \tag{6.28}$$

with $\varepsilon' = \pm 1$. These relations (note the difference with Eq. (6.14) for the strongly degenerate subcase A1) lead to a solution which is also a generalization of the Taub-NUT instanton, again supporting a self-dual axidilaton,

$$\mathrm{e}^{2\phi} = 1 + 2D\tau\,, \quad \kappa = \varepsilon'(1 - \mathrm{e}^{-2\phi})\,. \tag{6.29}$$

This solution is regular for positive τ provided both M and D are positive, leading again to a positive action (6.18).

B. Weakly degenerate case

This is the generic case $\boldsymbol{\lambda} \neq 0$, $\boldsymbol{\lambda}^2 = 0$ corresponding to $\operatorname{rank} B = 3$ and $B^3 \neq 0$, however $B^4 = 0$ since $\boldsymbol{\lambda}^2 = 0$. The expression for M includes three powers of τ

$$M = \eta\,(I + B\tau + B^2\tau^2/2 + B^3\tau^3/6)\,. \tag{6.30}$$

Because of this cubic behavior of the matrix representative, the evaluation of the action (5.43) is delicate owing to the occurence of infrared divergences in the individual factors, but leads directly to a finite result when the form (5.48) is used. The function Δ in (5.45) is then a polynomial of maximum degree 6 which is dominated for $\tau \to \infty$ by its leading term, $\Delta \sim O(\tau^p)$ ($p \leq 6$), leading to $\dot{\Delta}/\Delta \propto \tau^{-1}$ for $\tau \to \infty$. On the other hand, from the ALF behaviors (5.15), $\dot{\Delta}/\Delta = 2(M + D)$ for $\tau = 0$, leading to the same finite value for the boundary action

$$S_{\mathrm{inst}} = \frac{\beta}{2}(M - |N| + D) \tag{6.31}$$

as in the case of strongly degenerate ALF instantons.

Since the six charges are now related by the two conditions (5.23) and (6.3), the target space coordinates are generally given in terms of four independent charges. The relations between the charges are generically nonlinear, except in the following two subcases 1) and 2) where these relations linearize, leading to solutions depending on only three charges. By virtue of their orthogonality to the lighlike vector $\boldsymbol{\lambda}$, the vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are spacelike, so that all weakly degenerate solutions can be generated by $SO(2,1)$ transformations from either representative 1) or 2).

1) The relations

$$N = \varepsilon' M, A = \varepsilon' D, P = \varepsilon Q\,, \tag{6.32}$$

with $\varepsilon' = \pm 1$ independently of ε, obviously solve Eqs. (6.3) and (5.23). These relations (note again the difference with Eq. (6.14) for the strongly degenerate subcase A1)) generalize the relations (6.28) defining the neutral solution of case A2) and again lead to a generalization of the Taub-NUT instanton, with again the action (6.18). The vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are given by

$$\boldsymbol{\mu} = (\varepsilon_1(M-D), -\sqrt{2}Q, M-D)\,,\ \boldsymbol{\nu} = (M+D, -\varepsilon_1\varepsilon_2\sqrt{2}Q, \varepsilon_1(M+D))\,, \tag{6.33}$$

with $\varepsilon_1 = \varepsilon'$, $\varepsilon_2 = -\varepsilon\varepsilon'$. The target space coordinates read

$$\begin{aligned}
&f^{-1} = 1 + 2M\tau + 4(1+\varepsilon_2)DQ^2\tau^3/3\,,\\
&\chi = \varepsilon_1(f-1)\,,\\
&\mathrm{e}^{2\phi} = 1 + 2D\tau + 4(1-\varepsilon_2)MQ^2\tau^3/3 - f^{-1}v^2\,,\\
&\kappa = \varepsilon_1\{1 - [1-(1+\varepsilon_2)\sqrt{2}Qv\tau]\mathrm{e}^{-2\phi}\}\,,\\
&v = \sqrt{2}Qf\tau[1+(1+\varepsilon_2)D\tau + (1-\varepsilon_2)M\tau]\,,\\
&u = \varepsilon_1[v - (1+\varepsilon_2)\sqrt{2}Qf\tau]\,.
\end{aligned} \tag{6.34}$$

In the case $\varepsilon_2 = -1$ ($\varepsilon' = \varepsilon$), the relations (6.34) simplify to

$$\begin{aligned}
f &= 1 + \varepsilon_1\chi = (1+2M\tau)^{-1}, \quad v = \varepsilon_1 u = \sqrt{2}Q\tau\,,\\
\mathrm{e}^{2\phi} &= 1 + 2D\tau - 2Q^2\tau^2 - \frac{4}{3}MQ^2\tau^3\,, \quad \kappa = \varepsilon_1[1-\mathrm{e}^{-2\phi}]\,,
\end{aligned} \tag{6.35}$$

so that again the Maxwell and axidilaton fields are separately self-dual, leading for $M > 0$ to a regular metric which is that of the Taub-NUT instanton. However, the associated dilaton becomes singular at a finite distance from the centers $\tau \to \infty$.

In the case $\varepsilon_2 = +1$ ($\varepsilon' = -\varepsilon$), the metric is, on account of (5.20),

$$\mathrm{d}s^2 = f(r)(\mathrm{d}t - 2N\cos\theta\mathrm{d}\varphi)^2 + f^{-1}(r)\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right]\,, \tag{6.36}$$

with

$$f^{-1}(r) = 1 + \frac{2M}{r} + \frac{8DQ^2}{3r^3}\,. \tag{6.37}$$

If $DQ^2 > 0$ and $-M^3 < 9D/4Q^2$, the metric (6.36) is regular for $r > 0$, and is actually geodesically complete, as can be checked by the radial coordinate transformation $r = (8DQ^2/3)\rho^{-2}$, leading to the behavior

$$\mathrm{d}s^2 \simeq \left(\frac{8DQ^2}{3}\right)^2 \rho^{-6}(\mathrm{d}t - 2N\cos\theta\mathrm{d}\varphi)^2 + 4\mathrm{d}\rho^2 + \rho^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \tag{6.38}$$

near $\rho \to \infty$ ($r \to 0$). Thus the spacetime (6.36) is a wormhole interpolating between the two asymptotically flat regions $r \to 0$ and $r \to \infty$ where the curvature invariants

$$\begin{aligned} R &= -\frac{144DQ^2 r[2DQ^2+3r^2(M+r)]}{[8DQ^2+3r^2(2M+r)]^3}\,, \\ R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} &= \frac{2592r^2}{[8DQ^2+3r^2(2M+r)]^6}\,[608D^4Q^8-32D^3Q^6r^2(5M+24r) \\ &\quad +24D^2Q^4r^4(37M^2+40Mr+30r^2)+180DMQ^2r^8+27M^2r^{10}] \end{aligned} \tag{6.39}$$

(6.40)

vanish. The dilaton again develops a singularity at a finite distance.

2) The relations

$$N=[\varepsilon(D-M)-\varepsilon'\sqrt{2}Q]/2, \quad A=[\varepsilon(D-M)+\varepsilon'\sqrt{2}Q]/2, \quad \sqrt{2}P=\varepsilon'(M+D) \tag{6.41}$$

provide another, less obvious solution to Eqs. (6.3) and (5.23). The corresponding vectors $\boldsymbol{\mu},\boldsymbol{\nu}$ are given by

$$\boldsymbol{\mu}=(-\varepsilon'\sqrt{2}Q,\,-\sqrt{2}Q,\,M-D)\,, \quad \boldsymbol{\nu}=(M+D,\,\varepsilon'(M+D),\,-\varepsilon(M-D))\,. \tag{6.42}$$

The target space coordinates read

$$\begin{aligned} & f^{-1}=1+2M\tau+\alpha\beta\tau^2(1+\beta\tau/3)\,, \\ & \chi=\varepsilon\{1-f[1+\alpha\tau(1+\beta\tau)]\}\,, \\ & \mathrm{e}^{2\phi}=2[1+(M+D)\tau]-f^{-1}(1+v^2)\,, \\ & \kappa=-\varepsilon\mathrm{e}^{-2\phi}[1+(\alpha+2\beta)\tau-f^{-1}(1-uv)]\,, \\ & v=\varepsilon\varepsilon'[1-f(1+\alpha\tau)(1+\beta\tau)]\,, \\ & u=\varepsilon'[1-f(1+\beta\tau)]\,, \end{aligned} \tag{6.43}$$

with

$$\alpha \equiv M+D-\varepsilon\varepsilon'\sqrt{2}Q\,, \quad \beta \equiv M-D\,. \tag{6.44}$$

As in the case of the representative 1, dualization again leads to a metric of the form (6.36), with $f^{-1}(r)$ a cubic function of $\tau=1/r$ for $D \neq M$ (the solution with $D=M$ belongs to the strongly degenerate subcase 2), corresponding to a geodesically complete wormhole spacetime.

C. Non-degenerate case

In the case $\det B \neq 0$, the matrix B is no longer nilpotent, so that the matrix exponential in (5.11) does not reduce to a polynomial in τ. In order to evaluate it, we will make use of the Lagrange formula

$$\mathrm{e}^{B\tau} = \sum_{k=1}^{4} \mathrm{e}^{p_k\tau} \prod_{j\neq k} \frac{B-p_j}{p_k-p_j}\,,$$

where p_j are the eigenvalues of B, which from (5.35) are the four roots of $-\det B = 4\boldsymbol{\lambda}^2$. Contrary to the case of Lorentzian EMDA, the $SO(2,1)$ norm of the vector $\boldsymbol{\lambda}$ is indefinite and so $\det B$ may be positive or negative.

1) $\det B < 0$. It is convenient to normalize τ so that $\det B = -1$ (the general case may be recovered by a rescaling of the charges and an inverse rescaling of τ). The eigenvalues of B are $p_j = \pm 1$, $\pm\mathrm{i}$, leading to

$$2\mathrm{e}^{B\tau} = (\cosh\tau + \cos\tau)I + (\sinh\tau + \sin\tau)B + (\cosh\tau - \cos\tau)B^2 + (\sinh\tau - \sin\tau)B^3\,. \tag{6.45}$$

The corresponding target space coordinates are

$$\begin{aligned}
f^{-1} &= [1/2+G_1]\cosh\tau + [1/2-G_1]\cos\tau + [M+H_1(d_-,d_+,M)]\sinh\tau + [M-H_1(d_-,d_+,M)]\sin\tau\,,\\
f^{-1}\chi &= -[N-H_1(d_-,-d_+,N)]\sinh\tau - [N+H_1(d_-,-d_+,N)]\sin\tau\,,\\
f^{-1}v &= G_{2+}(\cosh\tau-\cos\tau) + [Q/\sqrt{2}+H_{2+}]\sinh\tau + [Q/\sqrt{2}-H_{2+}]\sin\tau\,,\\
f^{-1}u &= G_{2-}(\cosh\tau-\cos\tau) + [P/\sqrt{2}-H_{2-}]\sinh\tau + [P/\sqrt{2}+H_{2-}]\sin\tau\,,\\
\mathrm{e}^{2\phi} &= [1/2-G_1]\cosh\tau + [1/2+G_1]\cos\tau + [D+H_1(m_+,m_-,D)]\sinh\tau + [D-H_1(m_+,m_-,D)]\sin\tau - f^{-1}v^2\,,\\
\kappa\mathrm{e}^{2\phi} &= [A-H_1(-m_+,m_-,A)]\sinh\tau + [A+H_1(-m_+,m_-,A)]\sin\tau - f^{-1}uv\,,
\end{aligned} \tag{6.46}$$

where we have defined

$$\begin{aligned}
&G_1 = m_+m_- - d_+d_-\,, \quad G_{2\pm} = \frac{1}{\sqrt{2}}[(m_+\pm d_-)q_+ \pm (m_-\pm d_+)q_-]\,,\\
&H_1(x,y,z)) = xq_+^2 + yq_-^2 - 4xyz\,,\\
&H_{2\pm} = \frac{1}{\sqrt{2}}[2m_+d_-q_+ \pm 2m_-d_+q_- \mp (q_+\pm q_-)q_+q_-]\,.
\end{aligned}$$

2) $\det B > 0$. Normalizing τ so that $\det B = +4$, the eigenvalues of B are $p_j = \pm(1\pm\mathrm{i})$, leading to [27]

$$2\mathrm{e}^{B\tau} = 2g_1 I + 2g_+B + g_2B^2 + g_-B^3\,, \tag{6.47}$$

with

$$g_1 = \cosh\tau\cos\tau\,, \quad g_2 = \sinh\tau\sin\tau\,, \quad 2g_\pm = \cosh\tau\sin\tau \pm \sinh\tau\cos\tau\,. \tag{6.48}$$

The target space coordinates are

$$
\begin{aligned}
f^{-1} &= g_1 + 2Mg_+ + G_1 g_2 + H_1(d_-, d_+, M) g_- \,, \\
f^{-1}\chi &= -2Ng_+ + H_1(d_-, -d_+, N) g_- \,, \\
f^{-1}v &= \sqrt{2} Q g_+ + G_{2+} g_2 + H_{2+} g_- \,, \\
f^{-1}u &= \sqrt{2} P g_+ + G_{2-} g_2 - H_{2-} g_- \,, \\
\mathrm{e}^{2\phi} &= g_1 + 2Dg_+ - G_1 g_2 + H_1(m_+, m_-, D) g_- - f^{-1}v^2 \,, \\
\kappa \mathrm{e}^{2\phi} &= 2Ag_+ - H_1(-m_+, m_-, A) g_- - f^{-1}uv \,.
\end{aligned}
\tag{6.49}
$$

In both the above solutions the functions f^{-1} and $\mathrm{e}^{2\phi}$ oscillate and have an infinite number of simple roots for generic values of the parameters. It is easy to show that the roots τ_i of the scale factor f^{-1} mark curvature singularities through which the geodesics cannot be prolongated. Thus the physical solution must either lie in the interval $(0 < \tau < \tau_1)$ between the infinity $\tau = 0$ and the lowest root, or between two neighboring roots, $(\tau_i < \tau < \tau_{i+1})$. Only in the first case the solution is ALF and extremal by construction, so we can choose it as candidate instanton. However the corresponding on-shell action, given by (5.48) with the upper limit $\tau = \infty$ replaced by $\tau = \tau_1$, is divergent. This solution therefore cannot be accepted as instanton. It is interesting to note that, although it saturates the asymptotic no-force bound, it is not supersymmetric.

VII. DISCUSSION OF THE SOLUTIONS: EXCEPTIONAL ASYMPTOTICS

In this section we shall present the most relevant examples of null geodesic solutions with the various exceptional asymptotics behaviors outlined at the end of Sect. 4, without entering into a detailed discussion of all the possible solutions.

A. Case E1

In this ALE case, the natural background is flat four-dimensional Euclidean space [33], with $f_0 = \tau^{-1} = r$. The regularized trace $[k]$ of the extrinsic curvature of $\partial\mathscr{E}$ again vanishes, so that the net action $S_{\text{inst}} = S_1 + S_2$ is now given by

$$
S_{\text{inst}} = \frac{\beta}{4}\left(f_0^{-1} \dot{f}_0 \Big|_{\tau=0} - \left[\frac{\dot{\Delta}}{\Delta}\right]_{\tau=0}^{\tau=\infty} \right) = \frac{\beta}{2}\left(\dot{\phi}(0) - \frac{1}{2}\frac{\dot{\Delta}}{\Delta}\Big|_{\tau=\infty} \right) \,,
\tag{7.1}
$$

where we have used the ALE condition $\lim_{\tau\to 0}(f - f_0) = 0$. For all degenerate solutions, the contribution of the second term in (7.1) will vanish just as in the ALF case, so that the instanton action will simply

be proportional to the dilaton charge. The matrix B is now replaced by B_1' given in (B.15). The dualized one-forms are given by (5.19) with $-b$ replaced by the lower left-hand block of B_1', leading to

$$\omega = -(B_1')_{31}\cos\theta \mathrm{d}\varphi = \mp 2m_{\mp}\cos\theta \mathrm{d}\varphi \tag{7.2}$$

for $\tau = 1/r$.

We first discuss the strongly degenerate case. Applying the transformation (5.13), with the transformation matrix K given by (B.12), to the matrix (6.5) leads to the target space potentials for the ALE asymptotics:

$$\begin{aligned} f^{-1} &= 2m_{\mp}\tau\,, \quad \chi = \mp f\,, \\ v &= \frac{q_{\pm}}{2m_{\mp}}\,, \quad u = \pm v\,, \\ \mathrm{e}^{2\phi} &= 1 + 2D\tau - f^{-1}v^2\,, \quad \kappa = \pm(1 - \mathrm{e}^{-2\phi} - 2d_{\mp}\tau \mathrm{e}^{-2\phi})\,. \end{aligned} \tag{7.3}$$

The constant electromagnetic potentials can be gauged away to $q_{\pm} = 0$, implying $m_{\pm} = 0$. For the choice $M = 1/4$ (consistent with the ALE normalisation (4.18)), this leads to the solutions

$$f = \mp\chi = \tau^{-1}\,, \quad \mathrm{e}^{2\phi} = 1 + 2D\tau\,, \quad \kappa = \pm(1 - \mathrm{e}^{-2\phi})\,, \tag{7.4}$$

in the subcase 1, and

$$f = \mp\chi = \tau^{-1}\,, \quad \mathrm{e}^{2\phi} = 1 + 2D\tau\,, \quad \kappa = \mp(1 - \mathrm{e}^{-2\phi})\,, \tag{7.5}$$

in the subcase 2. These are the extremal dilato-axionic instantons [6] with self-dual scalar fields on a flat four-dimensional metric, a prototype of D-instantons. The generalisation to a multicenter harmonic function

$$\tau = \sum_{i=1}^{s} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \tag{7.6}$$

leads to non-trivial instanton solutions of the gravitating dilaton-axion system with a regular metric, namely flat space (4.20) for $s = 1$, the Eguchi-Hanson metric for $s = 2$ and lens spaces for higher s [3].

The weakly degenerate case leads to dyonic ALE instantons, generalizing the above solutions. Simple solutions can be obtained in the case of the representative 1. In the subcase $\varepsilon_2 = -1$, $\varepsilon_1 = \mp 1$ ($\varepsilon = \varepsilon' = \mp 1$), one obtains for the choice $M = 1/4$

$$\begin{aligned} f &= \mp\chi = \tau^{-1}\,, \quad v = \mp u = Q\tau\,, \\ \mathrm{e}^{2\phi} &= 1 + 2D\tau - \frac{Q^2}{3}\tau^3\,, \quad \kappa = \mp(1 - \mathrm{e}^{-2\phi})\,. \end{aligned} \tag{7.7}$$

For the boundary action (5.48) one obtains, as in the case of the strongly degenerate dilaton-axion instanton,

$$S_{\mathrm{inst}} = \frac{\beta D}{2} = 2\pi D\,, \tag{7.8}$$

with $\beta = 4\pi$ the period of the angular coordinate η, consistent with Rey's Bogomolnyi result [6].

In the subcase $\varepsilon_2 = +1$, $\varepsilon_1 = \mp 1$ ($\varepsilon = -\varepsilon' = \pm 1$), one obtains

$$\begin{aligned} f^{-1} &= 4M\tau + \frac{16}{3}DQ^2\tau^3\,, \quad \chi = \mp f\,, \\ v &= 2Q\tau f(1+2D\tau)\,, \quad u = \pm 2Q\tau f(1-2D\tau)\,, \\ \mathrm{e}^{2\phi} &= \tau f(1+2D\tau)\left(4M - 4Q^2\tau - \frac{8}{3}DQ^2\tau^2\right)\,, \\ \kappa &= \mp 4\tau^2 f\mathrm{e}^{-2\phi}\left(2MD + Q^2 - \frac{4}{3}D^2Q^2\tau^2\right)\,. \end{aligned} \tag{7.9}$$

For $M = 1/4$ the metric, of the form (6.36), is a wormhole interpolating between an ALE behavior for $r \to \infty$ and the conical ALF behavior (6.38) for $r \to 0$ ($\rho \to \infty$). At spatial infinity, the dilaton behaves as

$$\phi \simeq 1 + 2(D - 2Q^2)\tau \quad (\tau \to 0)\,, \tag{7.10}$$

and the action is given by (7.15) where D is replaced by that the effective dilaton charge $D - 2Q^2$. In both this and the preceding subcase, the dilaton develops a singularity at a finite distance. The exceptional, non ALE possibility $M = 0$ leads to a negative definite $\mathrm{e}^{2\phi}$.

B. Case E2

In the strongly degenerate case, transforming (6.5) by (5.13) with the transformation matrix (B.18) leads to the exceptional ALF solution

$$\begin{aligned} f^{-1} &= 1 + 2M\tau\,, \quad \chi = \frac{N}{M}(f-1)\,, \\ v &= q_\mp \tau f\,, \quad u = \pm q_\pm \tau f\,, \\ \mathrm{e}^{2\phi} &= \tau f\left[2d_\mp + (4Md_\mp - q_\mp^2)\tau\right]\,, \quad \kappa = \pm \mathrm{e}^{-2\phi} f\left[1 + 2M\tau - q_\pm q_\mp \tau^2\right]\,. \end{aligned} \tag{7.11}$$

If $d_\mp \neq 0$ ($d_\mp = 0$ leads to a negative definite $\mathrm{e}^{2\phi}$), the three-form associated with the axion field is, asymptotically,

$$H \simeq d_\mp\,(\mathrm{d}t \wedge \mathrm{d}\theta \wedge \sin\theta\,\mathrm{d}\varphi) \quad (r \to \infty)\,, \tag{7.12}$$

so that the one-form and three-form contributions to the action (2.1) are both linearly infra-red divergent. This divergence is similar to that of the bare (unregularized) purely gravitational action, suggesting that it can be regularized according to (2.16). This does not modify the value of the regularized action for the ALF instantons of Sect. 6 or for the ALE instantons of case E1, for which the background has a vanishing axion field. In the present case, we choose as background the solution (7.11) with $M = |N|$ (self-dual Taub-NUT

instanton metric) and $Q=P=0$. For this configuration, $\kappa_0=\pm e^{-2\phi_0}$, leading to $e^{4\phi_0}\kappa_0\dot\kappa_0=-2\dot\phi_0$, so that the total regularized action becomes

$$S_{\rm inst}=\frac{\beta}{4}\left(\left[\frac{\dot f}{f}-2(\dot\phi-\dot\phi_0)\right]_{\tau=0}^{\tau=\infty}-2|N|\right). \tag{7.13}$$

For the solution (7.11),

$$\dot\phi\simeq\frac{1}{2\tau}-\frac{q_\mp^2}{4d_\mp}\quad(\tau\to0). \tag{7.14}$$

while $\dot\phi(\infty)=0$, leading to the value of the boundary action

$$S_{\rm inst}=\frac{\beta}{2}\left[M-|N|-\frac{q_\mp^2}{4d_\mp}\right]. \tag{7.15}$$

In the subcase 1 with $\varepsilon=\mp1$, the solution

$$\begin{aligned}
f^{-1}&=1+2M\tau\,,\quad \chi=\pm(f-1)\,,\\
v&=2Q\tau f\,,\quad u=0\,,\\
e^{2\phi}&=4\tau f\left[D+(2MD-Q^2)\tau\right],\quad \kappa=\pm e^{-2\phi}\,,
\end{aligned}\tag{7.16}$$

is self-dual Taub-NUT supporting a purely electric field and a self-dual axidilaton. The dilaton field is regular provided $D\ge Q^2/2M$, however the action (7.15) with $d_\mp=2D$ is then negative unless $P=Q=0$. On the other hand, the subcase 2 can lead to regular instanton solutions with positive action. For instance, for the two-parameter family (6.26), $d_\mp=m_\mp$ and $q_\mp^2=2(M^2-N^2)$, leading to

$$e^{2\phi}=2m_\mp\tau f\left[1+m_\mp\tau\right], \tag{7.17}$$

which is positive definite if $m_\mp>0$, and to the value of the action

$$S_{\rm inst}=\frac{\beta}{4}\left(M\mp N-2|N|\right). \tag{7.18}$$

A sufficient condition for this to be positive, irrespective of the sign of N, is $M>3|N|$.

In the weakly degenerate case, the representative 1 with $\varepsilon_2=-1$, $\varepsilon_1=\mp1$ leads to a solution which is also electric Taub-NUT, but with a singular dilaton,

$$\begin{aligned}
f^{-1}&=1+2M\tau\,,\quad \chi=\mp(f-1)\,,\\
v&=2Q\tau\,,\quad u=0\,,\\
e^{2\phi}&=4\tau\left(D-Q^2\tau-\frac{2}{3}MQ^2\tau^2\right),\quad \kappa=\pm e^{-2\phi}\,.
\end{aligned}\tag{7.19}$$

Because the weakly degenerate and strongly degenerate instanton solutions have (for $d_\mp \neq 0$) the same asymptotic behavior, the action is again given by (7.15) with $M = |N|$ and $d_\mp = 2D$, and is again negative. The representative 1 with $\varepsilon_2 = +1$, $\varepsilon_1 = \mp 1$ gives

$$\begin{aligned} f^{-1} &= 1 + 2M\tau + \frac{8}{3}DQ^3\tau^3\,, \quad \chi = \mp(f-1)\,, \\ v &= 4DQ\tau^2 f\,, \quad u = \pm 2Q\tau f\,, \\ \mathrm{e}^{2\phi} &= 4D\tau f\left(1 + 2M\tau - \frac{4}{3}DQ^2\tau^3\right)\,, \\ \kappa &= \pm f\mathrm{e}^{-2\phi}\left(1 + 2M\tau - \frac{16}{3}DQ^2\tau^3\right)\,, \end{aligned} \tag{7.20}$$

leading to a wormhole metric of the form (6.36), with vanishing regularized action

$$S_{\text{inst}} = 0\,. \tag{7.21}$$

An example of a weakly degenerate representative 2 solution with a positive action is obtained from (6.41) with $Q = 0$, $\varepsilon = \pm 1$, leading to $d_\mp = \mp\varepsilon' q_\mp/\sqrt{2} = m_\pm$, so that the action is again given by (7.18), leading to

$$S_{\text{inst}} = \frac{\beta}{8}\left(3M - D \pm 2|M - D|\right)\,. \tag{7.22}$$

C. Case E3a

The strongly degenerate case leads to the exceptional ALE solution

$$\begin{aligned} f^{-1} &= 2m_\mp\tau\,, \quad \chi = \mp f\,, \\ v &= \frac{q_\pm}{\sqrt{2}m_\mp}\,, \quad u = 0\,, \\ \mathrm{e}^{2\phi} &= 2\left(d_\pm - \frac{q_\pm^2}{2m_\mp}\right)\tau\,, \\ \kappa &= \mp\mathrm{e}^{-2\phi}\,. \end{aligned} \tag{7.23}$$

The constant electric field can be gauged away to $q_\pm = 0$. The regularized action is now, for degenerate solutions,

$$S = \frac{\beta}{2}\left[\dot{\phi} - \dot{\phi}_0\right]_{\tau=0}\,, \tag{7.24}$$

with the solution (7.23) itself as the only possible background, so that the action vanishes identically.

The weakly degenerate representative 1 with $N = \mp M$, $A = \mp D$, $P = \pm Q$ (the other possibilities lead to vanishing f^{-1} or $e^{2\phi}$) leads to

$$\begin{aligned} f^{-1} &= 4M\tau + \frac{16}{3}DQ^2\tau^3\,, \quad \chi = \mp f\,, \\ v &= 2\sqrt{2}Q\tau f\,, \quad u = \mp 4\sqrt{2}DQ\tau^2 f\,, \\ \mathrm{e}^{2\phi} &= -8Q^2\tau^2 f\,, \\ \kappa &= \mp f \mathrm{e}^{-2\phi}\left(4M\tau - \frac{32}{3}DQ^2\tau^3\right). \end{aligned} \tag{7.25}$$

However, the dilaton field is negative definite. The regularized action (with $D = 0$ as background) again vanishes.

D. Case E3b

We shall discuss only the two limiting cases $\cos v = \pm 1$ (exceptional ALE) and $\sin v = \pm 1$ (magnetic linear dilaton):

1) $\underline{\cos v = \pm 1}$. In the strongly degenerate case we obtain

$$\begin{aligned} f^{-1} &= 2m_{\mp}\tau\,, \quad \chi = \mp f\,, \\ v &= 0\,, \quad u = \pm\frac{q_{\pm}}{\sqrt{2}m_{\mp}}\,, \\ \mathrm{e}^{2\phi} &= 2d_{\mp}\tau\,, \quad \kappa = \pm \mathrm{e}^{-2\phi}\,. \end{aligned} \tag{7.26}$$

The constant magnetic field can be gauged away to $q_{\pm} = 0$. The action (7.24) vanishes as in the case E3a.

The weakly degenerate representative 1 with $N = \mp M$, $A = \mp D$, $P = \pm Q$ leads, for the choice $M = 1/4$, to

$$\begin{aligned} f^{-1} &= \tau + \frac{16}{3}DQ^2\tau^3\,, \quad \chi = \mp f\,, \\ v &= 4\sqrt{2}DQ\tau^2 f\,, \quad u = \pm 2\sqrt{2}Q\tau f\,, \\ \mathrm{e}^{2\phi} &= 4D\tau^2 f\left(1 - \frac{8}{3}DQ^2\tau^2\right), \\ \kappa &= \pm f\mathrm{e}^{-2\phi}\left(\tau - \frac{32}{3}DQ^2\tau^3\right), \end{aligned} \tag{7.27}$$

while the representative 1 with $N = \mp M$, $A = \mp D$, $P = \mp Q$ leads (again for $M = 1/4$) to

$$\begin{aligned} f^{-1} &= \tau\,, \quad \chi = \mp f\,, \\ v &= \sqrt{2}Q\tau\,, \quad u = 0\,, \\ \mathrm{e}^{2\phi} &= 4D\tau - \frac{2}{3}Q^2\tau^3\,, \quad \kappa = \pm\mathrm{e}^{-2\phi}\,. \end{aligned} \tag{7.28}$$

In both cases, the action again vanishes.

2) $\underline{\sin\nu = \pm 1}$. In the strongly degenerate case,

$$
\begin{aligned}
& f^{-1} = (M + D \mp \sqrt{2}P)\tau\,, \quad \chi = -\frac{N+A}{M+D\mp\sqrt{2}P}\,, \\
& \nu = \pm\frac{N-A\pm\sqrt{2}Q}{M+D\mp\sqrt{2}P}\,, \quad u = \pm\frac{1+(M-D)\tau}{(M+D\mp\sqrt{2}P)\tau}\,, \\
& e^{2\phi} = \frac{(M+D\mp\sqrt{2}P)^2-(N-A\pm\sqrt{2}Q)^2}{M+D\mp\sqrt{2}P}\,\tau\,, \\
& \kappa = -\frac{(N-A\pm\sqrt{2}Q)(1+(M-D)\tau)-(M+D\mp\sqrt{2}P)(N+A)\tau}{[(M+D\mp\sqrt{2}P)^2-(N-A\pm\sqrt{2}Q)^2]\,\tau}\,.
\end{aligned}
\tag{7.29}
$$

The constant field χ can be gauged to zero by the choice $A = -N$. After taking into account the null geodesic and strong degeneracy conditions (5.23) and (6.21) ((6.14) leads to $e^{2\phi} = 0$), there remains the solution

$$
\begin{aligned}
& f = \pm u = \frac{1}{4M\tau}\,, \quad \chi = 0\,, \quad \nu = \pm\frac{N}{M}\,, \\
& e^{-2\phi} = \frac{M}{4(M^2-N^2)\tau}\,, \quad \kappa = -\frac{N}{M}e^{-2\phi}\,,
\end{aligned}
\tag{7.30}
$$

with vanishing regularized action (again, the only natural background is the solution itself). The weakly degenerate case leads to complicated expressions which we shall not give here.

VIII. MULTIPLE HARMONIC FUNCTIONS

The instantons listed above were incorporating only one independent harmonic function (including the multicenter solution in which all centers have equal charges). However, the construction (5.7) may be generalized [25, 27] to the case of several truly independent harmonic functions τ_a, $\Delta\tau_a = 0$, by replacing the exponent in (5.7) by a linear superposition

$$
M = A\exp\left(\sum_a B_a\tau_a\right). \tag{8.1}
$$

This solves the field equations (4.15) provided that the commutators $[B_a, B_b]$ commute with the B_c (for the proof see [27]):

$$
[[B_a, B_b], B_c] = 0\,. \tag{8.2}
$$

The three-dimensional Einstein equations (5.2) generalize to

$$
R_{ij} = \frac{1}{4}\sum_a\sum_b \mathrm{tr}\,(B_aB_b)\,\nabla_i\tau_a\nabla_j\tau_b\,, \tag{8.3}
$$

so that the three-space is Ricci flat if the matrices B_a satisfy

$$\mathrm{tr}\,(B_a B_b) = 0\,. \tag{8.4}$$

It follows from the above that the number of independent harmonic functions on which an extremal solution of the form (8.1) may depend is limited by the number of independent mutually orthogonal null vectors of the target space. As discussed in [27], for a locally Minkowskian target space with signature $(+p, -q)$ the maximum number of independent null vectors is $\inf(p,q)$. So in the present case of Euclidean EMDA, BPS solutions depending on three harmonic functions (as opposed to only two for Lorentzian EMDA [27]) are possible in principle.

We show in Appendix D that, in the case of Euclidean EMDA, the double commutation relations (8.2) together with the Ricci-flatness conditions (8.4) imply the apparently stronger commutation relations[2]

$$[B_a, B_b] = 0\,. \tag{8.5}$$

In that case, differentiation of (8.1) yields

$$M^{-1}\nabla M = \sum_a B_a \nabla \tau_a\,, \tag{8.6}$$

so that both the expressions (5.19) for the dualized one-forms and (5.48) for the boundary action generalize to linear superpositions.

Consider two matrices $B(\boldsymbol{\mu}, \boldsymbol{\nu})$ and $B'(\boldsymbol{\mu}', \boldsymbol{\nu}')$ of the form (5.28) where $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\mu}', \boldsymbol{\nu}'$ are any four $SO(2,1)$ vectors. Then

$$\mathrm{tr}\,(BB') = 4(\boldsymbol{\mu}\cdot\boldsymbol{\mu}' - \boldsymbol{\nu}\cdot\boldsymbol{\nu}'), \quad [B, B'] = 2\begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{B} & \mathscr{A} \end{pmatrix},$$

where

$$\begin{aligned}
\mathscr{A} &= (\boldsymbol{\mu}\wedge\boldsymbol{\mu}' - \boldsymbol{\nu}\wedge\boldsymbol{\nu}')^0 \sigma_2\,, \\
\mathscr{B} &= (\boldsymbol{\mu}\cdot\boldsymbol{\nu}' - \boldsymbol{\nu}\cdot\boldsymbol{\mu}')\sigma_0 - (\boldsymbol{\mu}\wedge\boldsymbol{\mu}' - \boldsymbol{\nu}\wedge\boldsymbol{\nu}')^3\sigma_1 + (\boldsymbol{\mu}\wedge\boldsymbol{\mu}' - \boldsymbol{\nu}\wedge\boldsymbol{\nu}')^1\sigma_3\,.
\end{aligned}$$

Thus, the conditions $\mathrm{tr}\,B^2 = 0$, $\mathrm{tr}\,B'^2 = 0$, $\mathrm{tr}\,(BB') = 0$ and $[B, B'] = 0$ lead to the following four scalar equations and one vector equation

$$\boldsymbol{\mu}^2 = \boldsymbol{\nu}^2\,, \quad \boldsymbol{\mu}'^2 = \boldsymbol{\nu}'^2\,, \quad \boldsymbol{\mu}\cdot\boldsymbol{\mu}' = \boldsymbol{\nu}\cdot\boldsymbol{\nu}'\,, \tag{8.7}$$

$$\boldsymbol{\mu}\cdot\boldsymbol{\nu}' = \boldsymbol{\nu}\cdot\boldsymbol{\mu}'\,, \quad \boldsymbol{\mu}\wedge\boldsymbol{\mu}' = \boldsymbol{\nu}\wedge\boldsymbol{\nu}'\,. \tag{8.8}$$

[2] This is not the case e.g. for Lorentzian Einstein-Maxwell gravity, where the only linearly independent matrices satisfying (8.2) do not commute but anticommute [27].

The analysis of this system (see Appendix D) reveals that only degenerate matrices are allowed, non-degenerate matrices leading uniquely to one-potential solutions described in the previous section. There are three two-potential classes of solutions. The first, with two strongly degenerate subcase 1 generators such that $\boldsymbol{\mu}' \pm \boldsymbol{\nu}' = \boldsymbol{\mu} \pm \boldsymbol{\nu} = 0$ may be directly extended to three-potential solutions. In the second class, the two generators belong to the lightlike sector of the strongly degenerate subcase 1, with $\boldsymbol{\mu}' \pm \boldsymbol{\nu}' = \boldsymbol{\mu} \mp \boldsymbol{\nu} = 0$, $\boldsymbol{\mu}' \mp \boldsymbol{\nu}' \propto \boldsymbol{\mu} \pm \boldsymbol{\nu}$. In the third class, one generator B is weakly degenerate, and the other strongly degenerate generator B' is proportional to B^3. We discuss these three classes in turn.

A. Three-potential class

The product of any two matrices B_a with $\boldsymbol{\nu}_a = \pm\boldsymbol{\mu}_a \ \forall a$ is identically zero, so that (8.1) with any number of such matrices will lead to an extremal solution. However this number is limited by the number (three) of linearly independent vectors $\boldsymbol{\mu}_a$, leading to three-potential solutions. In the ALF case, these are

$$M = \eta\,[I + B_1(\boldsymbol{\mu}_1, \pm\boldsymbol{\mu}_1)\tau_1 + B_2(\boldsymbol{\mu}_2, \pm\boldsymbol{\mu}_2)\tau_2 + B_3(\boldsymbol{\mu}_3, \pm\boldsymbol{\mu}_3)\tau_3]\,, \tag{8.9}$$

with τ_1, τ_2, τ_3 three independent multimonopole harmonic potentials. One may choose the three vectors $\boldsymbol{\mu}_a$ to control independently the gravitational, dilato-axionic and electromagnetic fields,

$$\boldsymbol{\mu}_1 = (\pm M, 0, M)\,,\ \boldsymbol{\mu}_2 = (\pm D, 0, -D)\,,\ \boldsymbol{\mu}_3 = (0, -\sqrt{2}Q, 0)\,, \tag{8.10}$$

where the first two vectors are null and the third is spacelike, leading to a solution depending on the three charges M, D and Q,

$$\begin{aligned} f &= 1 \pm \chi = (1 + 2M\tau_1)^{-1}\,, \\ \mathrm{e}^{-2\phi} &= 1 \pm \kappa = [1 + 2D\tau_2 - 2Q^2 f(\tau_1)\tau_3^2]^{-1}\,, \\ v &= \mp u = \sqrt{2}Q f(\tau_1)\tau_3\,. \end{aligned} \tag{8.11}$$

The general solution, which generalizes the one-potential strongly degenerate subcase 1 solutions, is a linear superposition of arbitrarily centered self-dual Taub-NUT metrics. The self-dual scalar and Maxwell sectors are determined by independent charges D, Q and independent multimonopole harmonic functions. Eq. (8.11) may also be written

$$\Delta \equiv f^{-1}\mathrm{e}^{2\phi} = (1 + 2M\tau_1)(1 + 2D\tau_2) - 2Q^2\tau_3^2\,. \tag{8.12}$$

Assuming the harmonic potentials to be normalized so that $\tau_i = \tau + \rho_i$, where $\tau(r) = 1/r$ and the ρ_i contain only higher harmonics, we find again the action to be given by (6.11).

Eq. (8.12) shows that the dilaton field will unavoidably develop singularities near the centers of τ_3, unless these are also centers of τ_1 and τ_2. Thus for dilaton regularity one must have $\tau_1 = \tau_3 + \tau_1'$, $\tau_2 = \tau_3 + \tau_2'$, where τ_1', τ_2' and τ_3 are three independent multicenter harmonic functions (6.20). In terms of these new potentials, the three-potential extremal solution

$$\begin{aligned} & f^{-1} = 1 + 2M(\tau_1' + \tau_3)\,, \quad f^{-1}v = \sqrt{2}Q\tau_3\,, \\ & f^{-1}\mathrm{e}^{2\phi} = (1+2M\tau_1')(1+2D\tau_2') + 2[M+D+2MD(\tau_1'+\tau_2')]\tau_3 + 2(2MD-Q^2)\tau_3^2 \end{aligned} \tag{8.13}$$

(together with the corresponding dual fields) is regular for $M \geq 0$, $D \geq 0$ and $Q^2 \leq 2MD$.

From these ALF solutions, three-potential solutions with exceptional asymptotics (ALE or cases E2, E3a or E3b) may be generated via the transformations (5.13) with the appropriate transformation matrices K given in Appendix B. The general three-potential ALE solution is, for $M = 1/4$,

$$\begin{aligned} f^{-1} &= \tau_1\,, \quad \chi = \mp f\,, \\ v &= 2Q\frac{\tau_1}{\tau_3}\,, \quad u = \pm v\,, \\ \mathrm{e}^{2\phi} &= 1 + 2\frac{D\tau_1\tau_2 - 2Q^2\tau_3^2}{\tau_1}\,, \quad \kappa = \pm(1-\mathrm{e}^{-2\phi})\,. \end{aligned} \tag{8.14}$$

For a monopole potential $\tau_1 = 1/r$, this corresponds to independent multicenter self-dual Maxwell and axidilaton fields living on four-dimensional Euclidean space. The instanton action is again proportional to the net dilaton charge, while the net electric and magnetic charges vanish. Similar configurations also exist for a multicenter potential τ_1, with the Euclidean space replaced by Eguchi-Hanson or lens spaces. The three-potential exceptional ALF solution (case E2)

$$\begin{aligned} f^{-1} &= 1 + 2M\tau_1\,, \quad \chi = \pm(f-1)\,, \\ v &= 2Q\tau_3 f\,, \quad u = 0\,, \\ \mathrm{e}^{2\phi} &= 4f\left[D\tau_2 + (2MD\tau_1\tau_2 - Q^2\tau_3^2)\right]\,, \quad \kappa = \pm\mathrm{e}^{-2\phi}\,, \end{aligned} \tag{8.15}$$

is the natural generalization of (7.16), with again a generically negative action. In the cases E3a or E3b, where one-potential instantons yield a vanishing regularized action, three-potential instantons also lead to a vanishing total action. The proof goes as follows. The function $\Delta = f^{-1}\mathrm{e}^{2\phi}$ is generically quadratic in the harmonic potentials τ_i (for instance, $\Delta = 8(2MD\tau_1\tau_2 - Q^2\tau_3^2)$ in the case E3a). After linearizing the potentials around the background potential $\tau = 1/r$ according to $\tau_i = \tau + \rho_i$, we find

$$\frac{\dot{\Delta}}{\Delta} = 2r + O(r^2\rho_i)\,. \tag{8.16}$$

The monopole component $2r$ is cancelled by the background subtraction, so that the integrand of (5.43) will be given by the dipole component of the $\rho_i(\mathbf{r})$ (the higher multipole contributions vanish for $r \to \infty$). This is odd in $\mathbf{r}$, and so leads to a vanishing boundary action after integration on the outer boundary.

B. Strongly degenerate two-potential class (dipole instantons)

Choosing for definiteness the strongly degenerate subcase 1 matrix B such that $\boldsymbol{\nu} = \boldsymbol{\mu}$ (this can be changed to $\boldsymbol{\nu} = -\boldsymbol{\mu}$ by exchanging the two matrices B and B') with $\boldsymbol{\mu}$ lightlike, $\boldsymbol{\mu}^2 = 0$, the matrix elements of B' are related to those of B through $-\boldsymbol{\nu}' = \boldsymbol{\mu}' = c^{-1}\boldsymbol{\mu}$, with c an arbitrary constant. The solutions of this class thus depend on three parameters (two for the null vector $\boldsymbol{\mu}$, and c). Assuming the two harmonic potentials τ and τ' to be normalized to $\tau \simeq \tau' \simeq 1/r$ at infinity, we choose these parameters to be the net charges M, N, Q and P constrained by

$$2M^2 + Q^2 = 2N^2 + P^2\,, \tag{8.17}$$

and take

$$\boldsymbol{\nu} = \boldsymbol{\mu} = \left(\frac{PM+QN}{Q+P}, -\frac{Q-P}{\sqrt{2}}, \frac{QM+PN}{Q+P}\right), \quad -\boldsymbol{\nu}' = \boldsymbol{\mu}' = \left(\frac{PM+QN}{Q-P}, -\frac{Q+P}{\sqrt{2}}, \frac{QM+PN}{Q-P}\right). \tag{8.18}$$

Because $B^2 = BB' = B'^2 = 0$, the exponential (8.1) linearizes, and the equations (6.6) giving the target space potentials in the ALF case generalize to

$$\begin{aligned}
&f^{-1} = 1 + m_+\tau + m_-\tau'\,, \quad f^{-1}\chi = -m_+\tau + m_-\tau'\,,\\
&f^{-1}v = \frac{1}{\sqrt{2}}\left[q_-\tau + q_+\tau'\right], \quad f^{-1}u = \frac{1}{\sqrt{2}}\left[-q_-\tau + q_+\tau'\right],\\
&f^{-1}\mathrm{e}^{2\phi} = 1 + (m_+ + d_-)\tau + (m_- + d_+)\tau' + (m_+ + d_-)(m_- + d_+)\tau\tau'\\
&f^{-1}\kappa\mathrm{e}^{2\phi} = -d_-\tau + d_+\tau' + (m_+d_+ - m_-d_-)\tau\tau'\,,
\end{aligned} \tag{8.19}$$

where the dilato-axionic charges $d_\pm$ are related to the gravitational and electromagnetic charges by (6.3). The relations (8.17) together with (6.3) imply that the net charges again satisfy the balance condition (5.23).

We choose for the harmonic potentials τ and τ' two monopole potentials $1/|\mathbf{r} \pm \boldsymbol{a}|$. It is convenient to choose $\boldsymbol{a}$ directed along the z axis and to introduce prolate spheroidal coordinates (r, θ, φ) related to the the cartesian coordinates (x, y, z) by

$$x = \sqrt{r^2 - a^2}\sin\theta\cos\varphi, \quad y = \sqrt{r^2 - a^2}\sin\theta\sin\varphi, \quad z = r\cos\theta, \qquad (r \geq a)\,. \tag{8.20}$$

In these coordinates, the three-metric is

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \frac{r^2 - a^2\cos^2\theta}{r^2 - a^2}\mathrm{d}r^2 + (r^2 - a^2\cos^2\theta)\mathrm{d}\theta^2 + (r^2 - a^2)\sin^2\theta\mathrm{d}\varphi^2\,, \tag{8.21}$$

and the harmonic potentials are

$$\tau = \frac{1}{r + a\cos\theta}\,, \quad \tau' = \frac{1}{r - a\cos\theta}\,. \tag{8.22}$$

The four-dimensional metric

$$ds^2 = \frac{r^2 - a^2\cos^2\theta}{\Sigma}(\mathrm{d}t - \omega_\varphi \mathrm{d}\varphi)^2 + \Sigma\left[\frac{\mathrm{d}r^2}{r^2 - a^2} + \mathrm{d}\theta^2 + \frac{(r^2 - a^2)\sin^2\theta}{r^2 - a^2\cos^2\theta}\mathrm{d}\varphi^2\right], \tag{8.23}$$

with

$$\omega_\varphi = \frac{2}{r^2 - a^2\cos^2\theta}\left[N(r^2 - a^2)\cos\theta + aMr\sin^2\theta\right], \quad \Sigma = r^2 + 2Mr - 2Na\cos\theta - a^2\cos^2\theta, \tag{8.24}$$

is supported by the axidilaton and electromagnetic potentials

$$\begin{aligned} e^{2\phi} &= 1 + 2\Sigma^{-1}\left[D(r+M) + A(N + a\cos\theta) + M^2 - N^2\right], \\ \kappa &= 2e^{-2\phi}\Sigma^{-1}\left[A(r+M) + D(N + a\cos\theta)\right], \\ v &= \sqrt{2}\Sigma^{-1}(Qr + aP\cos\theta), \quad u = \sqrt{2}\Sigma^{-1}(Pr + aQ\cos\theta). \end{aligned} \tag{8.25}$$

This solution — the Euclidean counterpart to the Lorentzian rotating extremal Taub-NUT dyon of [27] — has again finite action (6.11). The metric is regular for $|N| \leq M$ (Σ cannot vanish in this case, owing to $r \geq a$). The dilaton field is then also regular ($D^2 - A^2 = M^2 - N^2$ leads to $|A| \leq D$, so that $\Sigma e^{-2\phi} \geq 0$ for $r \geq a$). In the limit $M = \pm N$, $P = \mp Q$, $A = \mp D$ one recovers the one-potential solutions of the strongly degenerate subcase 1b. A more interesting limiting case is $N = A = Q = 0$, leading to a magnetic solution with monopole charges $M = D = \pm P/\sqrt{2}$ and dipole moments aM (gravitational), aD (axionic) and aP (electric). Finally, one can linearly superpose such solutions, replacing the harmonic potentials (8.22) by

$$\tau = \sum_{i=1}^{s}\frac{1}{|\mathbf{r} - \mathbf{r}_i + \boldsymbol{a}_i|}, \quad \tau' = \sum_{i=1}^{s}\frac{1}{|\mathbf{r} - \mathbf{r}_i - \boldsymbol{a}_i|}, \tag{8.26}$$

with arbitrary orientations and magnitudes of the dipoles $\boldsymbol{a}_i$.

Contrary to the three-potential case, dipole instantons with exceptional asymptotics are not possible. The reason is that the transformation matrices (B.12), (B.18), (B.28) involve a $\pm$ sign. Before applying the transformation (5.13) to the dipole ALF solution, one must choose a definite sign and thus a definite polarity, so that e.g. it is not possible to superpose self-dual and anti-self-dual ALE instantons.

C. Third class

The two-potential solutions of this class are of the form

$$M = \eta\left[I + B\tau + \frac{1}{2}B^2\tau^2 + \frac{1}{6}B^3(c\tau' + \tau^3)\right], \tag{8.27}$$

with B weakly degenerate. These can be treated similarly to the weakly degenerate solutions considered in Sect. 6B in the ALF case, and in Sect. 7 for the various exceptional asymptotics, so we do not repeat the analysis here.

IX. SIX-DIMENSIONAL INTERPRETATION

As shown in [52], EMDA can be regarded as a consistent truncation of six-dimensional vacuum general relativity (E6). If the six-dimensional metric with two commuting Killing vectors is parametrised by

$$\mathrm{d}s_6^2 = \mathrm{d}s_4^2 + \lambda_{ab}(\mathrm{d}x^a + \sqrt{2}A^a_\mu \mathrm{d}x^\mu)(\mathrm{d}x^b + \sqrt{2}A^b_\mu \mathrm{d}x^\mu)\,, \tag{9.1}$$

with eleven Kaluza-Klein matter fields λ_{ab} and A^a_μ $(a,b=5,6)$, the five covariant constraints [48]

$$\det(\lambda)=1\,,\quad F^a_{\mu\nu} = -\varepsilon^{ab}\lambda_{bc}\tilde{F}^c_{\mu\nu} \tag{9.2}$$

reduce the compactified E6 to EMDA, with only six matter fields.

This may be generalized to the case where both $g_{\mu\nu}$ and λ_{ab} have arbitrary signatures. Assume only

$$F^a_{\mu\nu} = \eta\varepsilon^{ab}\lambda_{bc}\tilde{F}^c_{\mu\nu}\,, \tag{9.3}$$

where $\eta=\pm1$ (actually, the sign of η is irrelevant). It follows from (9.3) that

$$\tilde{F}^a_{\mu\nu} = \eta\varepsilon^{ab}\lambda_{bc}\tilde{\tilde{F}}^c_{\mu\nu}\,. \tag{9.4}$$

If $g_{\mu\nu}$ is Lorentzian, $\tilde{\tilde{F}}^c_{\mu\nu} = -F^c_{\mu\nu}$, so that

$$F^a_{\mu\nu} = -\varepsilon^{ab}\lambda_{bc}\varepsilon^{cd}\lambda_{de}F^a_{\mu\nu} = \det(\lambda)F^a_{\mu\nu}\,, \tag{9.5}$$

implying $\det(\lambda)=+1$ (which is the integrability condition for (9.3). Conversely, if $g_{\mu\nu}$ is Euclidean, $\tilde{\tilde{F}}^c_{\mu\nu} = F^c_{\mu\nu}$, and $\det(\lambda)=-1$. It follows that the six-dimensional signature must in all cases be negative. Two cases will lead to Euclidean $(+++)$ signature after further reduction to three dimensions:

1) Six-dimensional signature $(---+++)$. The target space for E6 reduced to three dimensions is $SL(4,R)/SO(4) = SL(4,R)/(SO(3)\times SO(3))$. After reduction relative to two timelike Killing vectors $(--)$ and truncation, this leads to phantom EMDA with Lorentzian spacetime and target space (after reduction to three dimensions) $Sp(4,R)/(SO(3)\times SO(2))$. The embedding of $sp(4,R)$ in $sl(4,R)$ is treated in Appendix A of [27]. Using the conventions of that paper, the $SO(3)\times SO(3)$ is generated by

$$(K^0,\Gamma^1_0,\Gamma^2_0)+(\Sigma_0,\Gamma^0_1,\Gamma^0_2)\,, \tag{9.6}$$

while only the first four generators $(U_a = 1/2(\Sigma_0,\Gamma^0_1,K^0), U_2 = 1/2\Gamma^0_2))$ remain after truncation to phantom EMDA.

The other case has six-dimensional signature $(-+++++)$, with target space $SL(4,R)/SO(2,2) = SL(4,R)/(SO(2,1)\times SO(2,1))$. Two reductions to four dimensions (leaving at least three spacelike directions) are possible:

2) Reduction relative to two spacelike Killing vectors ($++$), leading to normal EMDA with Lorentzian spacetime and target space $Sp(4,R)/(SO(2,1)\times SO(2))$. Choosing for generators of the $SO(2,1)\times SO(2,1)$ the set

$$(\Sigma_0,\Sigma_1,\Sigma_2)+(K^0,K^1,K^2)\,, \tag{9.7}$$

the first four of these correspond to the generators (V_1,W_1,U_0,U_3) of $SO(2,1)\times SO(2)$.

3) Reduction relative to one timelike and one spacelike Killing vectors ($-+$). This leads after truncation to Euclidean EMDA with target space $Sp(4,R)/(SO(2,1)\times SO(1,1))$. Choosing for generators of the $SO(2,1)\times SO(2,1)$ the set

$$(\Gamma^2_0,\Sigma_1,\Gamma^2_2)+(\Gamma^1_1,\Gamma^0_1,K^2)\,, \tag{9.8}$$

the first four of these correspond to the generators (4.7) of $SO(2,1)\times SO(1,1)$.

Thus, reduction of six-dimensional Lorentzian gravity to four dimensions Euclidean EMDA is implemented by the constraints

$$F^a_{\mu\nu}=\pm\varepsilon^{ab}\lambda_{bc}\tilde{F}^c_{\mu\nu}\,,\quad \det(\lambda)=-1\,, \tag{9.9}$$

the last one being the integrability condition for the first six. In the following we will note $x^\mu=(\xi,r,\theta,\varphi)$ the four-dimensional Euclidean coordinates (with ξ the Euclidean time), and $x^a=(t,\eta)$ those of the two extra dimensions. The standard Euclidean EMDA parametrisation corresponds to the upper sign in (9.9) and

$$\begin{aligned}\lambda &= \begin{pmatrix} -\mathrm{e}^{-2\phi}+\kappa^2\mathrm{e}^{2\phi} & \kappa\mathrm{e}^{2\phi}\\ \kappa\mathrm{e}^{2\phi} & \mathrm{e}^{2\phi}\end{pmatrix},\\ A^a &= (A,\,B)\,,\quad F_{\mu\nu}(B)\equiv \mathrm{e}^{-2\phi}\tilde{F}_{\mu\nu}(A)-\kappa F_{\mu\nu}(A)\,.\end{aligned} \tag{9.10}$$

For instance, in the neutral case $P=Q=0$, the two-potential solution (8.11) uplifted to six dimensions leads (after putting $\mathrm{d}\eta'\equiv\pm\mathrm{d}\eta+\mathrm{d}t$) to the solution

$$\begin{aligned}\mathrm{d}s_6^2 &= -2\mathrm{d}\eta'\mathrm{d}t+\mathrm{e}^{2\phi}\mathrm{d}\eta'^2+f(\mathrm{d}\xi-\omega_i\mathrm{d}x^i)^2+f^{-1}\mathrm{d}\mathbf{x}^2\,,\\ \mathrm{e}^{2\phi} &= 1+2D\tau_2\,,\quad f^{-1}=1+2M\tau_1\,,\quad \nabla\wedge\boldsymbol{\omega}=\pm\nabla f^{-1}\,,\end{aligned} \tag{9.11}$$

in terms of two independent multicenter harmonic functions τ_1 and τ_2. This corresponds to a six-dimensional plane wave propagating on a four-dimensional multi-Euclidean Taub-NUT bulk. In the special case $\tau_1=0$, this solution reduces to

$$\mathrm{d}s_6^2=-2\mathrm{d}\eta'\mathrm{d}t+\mathrm{e}^{2\phi}\mathrm{d}\eta'^2+\mathrm{d}\xi^2+\mathrm{d}\mathbf{x}^2\,, \tag{9.12}$$

which is the direct product of the ξ axis by the multicenter “antigravitating” solution of five-dimensional vacuum general relativity found by Gibbons [53] (see also [25]). Similarly, the ALE two-potential solution (8.14) leads to a six-dimensional vacuum metric similar to (9.11) with $f^{-1} = 4M\tau_1$, corresponding to a flat, Eguchi-Hanson or lens-space four-dimensional bulk. In the monopole case $f = r = \rho^2/4$, we recover the multicenter metric with flat Euclidean four-dimensional bulk

$$ds_6^2 = -2\mathrm{d}\eta'\mathrm{d}t + \mathrm{e}^{2\phi}\mathrm{d}\eta'^2 + \mathrm{d}\mathbf{x_4^2}\,, \tag{9.13}$$

where $\mathrm{e}^{2\phi}$, from the footnote 1, is harmonic in the four-space. This is a special case of the uplift to six dimensions by (9.1) of four-dimensional dilato-axionic multi-instantons [8] with $\mathrm{e}^{2\phi}$ a generic harmonic function in four dimensions, previously given in [48].

The reduction (9.1) and the constraints (9.9) are invariant under $SL(2,R)$ transformations, so that in the six-dimensional context the fields $\mathrm{e}^{2\phi}$ and κ are defined only up to such transformations. This has two consequences:

1) Exceptional asymptotics of $\mathrm{e}^{2\phi}$ can be transformed to generic ($\phi(\infty) = 0$) asymptotics. If $\mathrm{e}^{2\phi} \sim O(\tau)$ for $\tau \to 0$, with e.g. $\kappa \sim -\mathrm{e}^{-2\phi}$, then

$$\lambda(\infty) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

This can be transformed to generic asymptotics $\lambda(\infty) = \mathrm{diag}(-1,1)$ by the linear transformation

$$\hat{\lambda} = A^T\lambda A\,, \quad A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{9.14}$$

leading to

$$\begin{aligned} \mathrm{e}^{2\hat{\phi}} &= \frac{1}{2}[(\kappa-1)^2\mathrm{e}^{2\phi} - \mathrm{e}^{-2\phi}] \simeq 1 + \frac{1}{2}\mathrm{e}^{2\phi}\,, \\ \hat{\kappa} &= \frac{1}{2}\mathrm{e}^{-2\hat{\phi}}[(\kappa^2-1)\mathrm{e}^{2\phi} - \mathrm{e}^{-2\phi}] \simeq \mathrm{e}^{-2\hat{\phi}} - 1\,. \end{aligned} \tag{9.15}$$

So exceptional asymptotics of $\mathrm{e}^{2\phi}$ do not lead to new six- dimensional solutions.

2) Zeroes of $\mathrm{e}^{2\phi}$ are not really relevant. The matrix λ remains regular at a zero of $\mathrm{e}^{2\phi}$ provided

$$\kappa\mathrm{e}^{2\phi} \simeq \pm 1 + O(\mathrm{e}^{2\phi}) \tag{9.16}$$

near such a zero; then, a transformation such as (9.14) restores a positive $\mathrm{e}^{2\phi}$. This is the case for the strongly degenerate solutions of type 1, and presumably also for the strongly degenerate solutions of type 2. This is also the case for the weakly degenerate representative 1 with $\varepsilon_2 = -1$. In the case of the ALF weakly degenerate representative 1 with $\varepsilon_2 = +1$,

$$\mathrm{e}^{2\phi} = f(1+2D\tau)(1+2M\tau - 2Q^2\tau^2 - \frac{4}{3}DQ^2\tau^3)\,. \tag{9.17}$$

For $\tau = -1/2D$, v vanishes, leading to $\kappa e^{2\phi} = -\varepsilon_1$, and the regularity condition is fulfilled. However, this does not seem to be the case for the other zeroes of $e^{2\phi}$, which would correspond to true singularities.

X. CONCLUSION

To summarize, we have presented a detailed investigation of extremal Euclidean solutions of EMDA theory (one-vector truncation of $D = 4, N = 4$ supergravity) using the purely bosonic technique of constructing extremal solutions as null geodesic curves of the three-dimensional sigma-model target space. This target space is the coset $Sp(4,\mathbb{R})/GL(2,\mathbb{R})$ which is yet another homogeneous space of the $Sp(4,\mathbb{R})$ U-duality group, apart from the previously discussed $Sp(4,\mathbb{R})/U(1,1)$ and $Sp(4,\mathbb{R})/U(2)$ corresponding to time-like and space-like reductions of Lorentzian EMDA. The new coset $Sp(4,\mathbb{R})/GL(2,\mathbb{R})$ is a six-dimensional homogeneous space with the signature $+++---$, and thus possesses three independent null directions. The Euclidean extremal solutions constitute various isotropic geodesic surfaces of this space which can be further classified according to the rank of the corresponding matrix generators. This purely bosonic classification is a priori not related with the classification of Killing spinors.

Though the derivation of the three-dimensional EMDA sigma model in the Lorentzian sector has been known for a long time, we had to reconsider it in the Euclidean case, taking into account previously ignored boundary terms arising in the dualizations involved. The bulk sigma-model action vanishes on-shell, so the instanton action is given entirely by the boundary terms. To get rid of infra-red divergences, which are generically present for non-compact spaces, we used the matched background subtraction method. For the three-dimensional boundary action we then obtained a very simple expression using the matrix formulation of the sigma model.

Dimensional reduction along a compactified time direction generically leads to solutions with ALF asymptotic structure. Instantons with exceptional asymptotics were shown to arise in the case of asymptotically vanishing (inverse) scale factor of this reduction (ALE instantons) or, in view of the intrinsic duality between the metric and axidilaton sectors [54], in the case of asymptotically vanishing exponentiated dilaton, or in the combination of these two cases. The coset matrices corresponding to different asymptotic behaviors were shown to be related by coset isometries.

Our classification scheme for extremal instantons refers to the algebraic nature of the corresponding matrix generators and involves the following types: i) nilpotent rank 2 (strongly degenerate), ii) nilpotent rank 3 (weakly degenerate) and iii) non-degenerate. Inside each type, further classification is provided according to the nature of the charge vectors. The solutions, most of which are new, include single-center and multi-center harmonic functions. The instanton action is finite for the classes i) and ii), independently

of the possible presence of singularities of the dilaton function at finite distance from the centers. The case iii) splits into two subcases depending on the sign of the determinant of the matrix generator, in both these subcases the scale factor and the dilaton function develop singularities at finite distance, and the instanton action is divergent. Although we did not investigate here Killing spinor equations, we believe that at least the strongly degenerate solutions are truly supersymmetric, while the non-degenerate are not (the weakly degenerate case requires further study).

The above classification relates to solutions generated by a single harmonic function (including the multi-center case). Our method also allows for the possibility of multiple independent harmonic functions. Considering the algebra of generators developing null geodesics of the target space, one generically finds the compatibility condition demanding vanishing of the triple commutators of generators. We have shown that in the case of Euclidean EMDA these imply a stronger condition of vanishing of all their pairwise commutators, which effectively linearizes the total generating current. Given the fact that the Euclidean sigma-model has three independent null directions (contrary to two in the Lorentzian case) we have shown that there exist solutions generated up to three independent harmonic functions. A first class of solutions is three-potential (all of which are possibly multicenter) and corresponds to a linear superposition of arbitrarily centered self-dual Taub-NUTs dressed with self-dual axidilaton and Maxwell fields. The second class is two-potential (dipole) and includes the Euclidean counterparts of the EMDA rotating extremal Taub-NUT and IWP solutions, while the third class, also two-potential, is built from a nilpotent matrix generator of rank 3 (weakly degenerate).

Apart from some simple extremal solutions, which were previously known explicitly in the purely scalar ALE sector [6], we were able to construct some new scalar ALF and ALE solutions, such as dilaton-axion dressed Taub-NUT, Eguchi-Hanson and lens-space instantons. We also found new types of solutions which are wormholes interpolating between ALF or ALE and conical ALF spaces. All electrically and magnetically charged solutions are entirely new except for those which were (or could be) found by euclideanization of known Lorentzian black hole and/or IWP-type solutions, which we rederived in our general treatment as well. The new charged ALE solutions found here include, among others, purely electric solutions, as well as purely magnetic instantons with linear dilaton asymptotics.

The last group of results consists in the six-dimensional uplifting of the four-dimensional EMDA instantons. Since the three-dimensional U-duality group $Sp(4,\mathbb{R})$ of EMDA is a subgroup of $SL(4,\mathbb{R})$, which is the U-duality group of vacuum six-dimensional gravity reduced to three dimensions, it is clear that any solution of four-dimensional EMDA of the type discussed in this paper can be interpreted as a solution of six-dimensional Einstein gravity without matter fields. We present details of this relationship and give the explicit six-dimensional form for some such instanton solutions. This uplift demonstrates, in particu-

lar, that zeroes of the dilaton exponent are not really relevant and can be resolved in the six-dimensional interpretation.

We have described extremal intantons using a purely bosonic method. Further work is needed to study the Killing spinor equations for Euclidean EMDA. We also believe that the new types of BPS instantons found here might give rise to new families of more general non-BPS EMDA instantons and wormholes, other methods are needed to study them. Another perspective of investigation of one-dimensional subspaces of the target space consists in further reduction of the sigma model to two-dimensions and application of Lax-pair integration techniques [55].

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Appendix A: Phantom EMDA

Similarly to the case of phantom Lorentzian EMDA, treated in [50], the action for phantom Euclidean EMDA, corresponding to a repulsive coupling of the electromagnetic field to gravity, is obtained from that of normal Euclidean EMDA by the analytical continuation $\phi \to \phi + \mathrm{i}\pi/2$. Kaluza-Klein reduction thus leads to the target space metric

$$\mathrm{d}l^2 = \frac{1}{2} f^{-2}\mathrm{d}f^2 - \frac{1}{2} f^{-2}(\mathrm{d}\chi + v\mathrm{d}u - u\mathrm{d}v)^2 - f^{-1}\mathrm{e}^{-2\phi}\mathrm{d}v^2 + f^{-1}\mathrm{e}^{2\phi}(\mathrm{d}u - \kappa\mathrm{d}v)^2 + 2\mathrm{d}\phi^2 - \frac{1}{2}\mathrm{e}^{4\phi}\mathrm{d}\kappa^2\,, \quad \text{(A.1)}$$

which has the same signature $(+3,-3)$ as the target space metric (3.20) of normal Euclidean EMDA, and thus corresponds to the same coset $Sp(4,\mathbb{R})/GL(2,\mathbb{R})$. However, analytical continuation of the matrix representative leads to a matrix representative $\overline{M}$ of the same form (4.9) as for normal EMDA, but with the 2×2 block matrices

$$\overline{P} = \mathrm{e}^{-2\phi}\begin{pmatrix} f\mathrm{e}^{2\phi} - v^2 & -v \\ -v & -1 \end{pmatrix}, \quad \overline{Q} = \begin{pmatrix} vw - \chi & w \\ w & -\kappa \end{pmatrix}, \qquad \text{(A.2)}$$

which coincide with those of normal Lorentzian EMDA. The antisymplectic matrix $\overline{M}$, with the asymptotic behavior

$$\eta = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} , \tag{A.3}$$

is related to that of of normal Euclidean EMDA by the $Sp(4,\mathbb{R})$ transformation (5.13), with

$$K = \frac{1}{2} \begin{pmatrix} \sigma_0+\sigma_3 & \sigma_0-\sigma_3 \\ -\sigma_0+\sigma_3 & \sigma_0+\sigma_3 \end{pmatrix} . \tag{A.4}$$

Appendix B: Exceptional asymptotic behaviors

The asymptotic solution of the sigma model field equations (4.15) for the matrix $M(\mathbf{r})$ given by (4.9) is

$$M(r) \simeq A(I+Br^{-1}) , \tag{B.1}$$

with A a constant symmetric antisymplectic matrix and B a constant symplectic matrix. The conditions on the 4×4 matrix A imply that it can be written in block form as

$$A = \begin{pmatrix} \alpha & \beta \\ \beta^T & \gamma \end{pmatrix} , \tag{B.2}$$

with the 2×2 matrices α, β and γ constrained by

$$\alpha^T = \alpha , \quad \gamma^T = \gamma , \tag{B.3}$$

$$\alpha\beta^T - \beta\alpha = 0 , \tag{B.4}$$

$$\beta^T\gamma - \gamma\beta = 0 , \tag{B.5}$$

$$\beta^2 - \alpha\gamma = 1 . \tag{B.6}$$

As discussed in Sect. 4, the generic matrix A may be gauge-transformed to η given by (4.8), corresponding to the ALF asymptotic behavior, except in the three exceptional cases 1) $f^{-1}(\infty)=0$, 2) $\mathrm{e}^{2\phi}(\infty)=0$, and 3) $f^{-1}(\infty)=\mathrm{e}^{2\phi}(\infty)=0$. We consider these three possibilities in turn.

1. $f^{-1}(\infty)=0$, $\mathrm{e}^{2\phi}(\infty)=1$.

From (B.1) f rises linearly as r, so the finiteness of M_{12} implies the asymptotic behavior $v\simeq ar+$ constant, resulting in $f^{-1}v^2 \sim ar$ which conflicts with the finiteness of M_{22} unless $a=0$. Thus $v(\infty)$ is

constant, and both ϕ and v may be gauge transformed to $\phi(\infty) = 0$, $v(\infty) = 0$, leading to

$$\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \tag{B.7}$$

The constraint (B.4) leads to $(f^{-1}u)(\infty) = 0$, leading to $u(\infty) =$ constant, which may again be gauge-transformed to 0. The matrix $\beta = (P^{-1}Q)(\infty)$ is

$$\beta = \begin{pmatrix} -(f^{-1}\chi)(\infty) & 0 \\ 0 & -\kappa(\infty) \end{pmatrix} . \tag{B.8}$$

The constant $\kappa(\infty)$ may be gauge-transformed to 0, while the constraint (B.6) leads to

$$\chi(r) \simeq \mp f(r) + c \tag{B.9}$$

asymptotically. Gauging the additive constant c to 0, we obtain $\beta = \pm(1-\alpha)$, and $\gamma = -\alpha$ from the last equation (4.11), so that finally

$$A = \eta'_1 \equiv \begin{pmatrix} 0 & 0 & \pm 1 & 0 \\ 0 & 1 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{B.10}$$

This is related to the matrix η of (4.8) by

$$\eta'_1 = K_1^T \eta K_1 , \tag{B.11}$$

with

$$K_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \pm\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \mp\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.12}$$

(actually there is a one-parameter family of such matrices K_1). The isotropy subagebra leaving invariant η'_1 is obtained from (2.44) by

$$\mathrm{lie}\,(H'_1) = K_1^{-1} \mathrm{lie}\,(H) K_1 \tag{B.13}$$

(in the present case, $K_1^{-1} = K_1^T$). So the null geodesics going through the point η'_1 are

$$M'_1 = \eta'_1 \mathrm{e}^{B'_1 \tau} = K_1^T \eta \, \mathrm{e}^{B\tau} K_1 = K_1^T M K_1 , \tag{B.14}$$

with

$$B_1' = K_1^{-1} B K_1 = \begin{pmatrix} 0 & -q_\mp & \pm 2m_\pm & \mp q_\mp \\ -q_\pm & 2D & \mp q_\mp & -2A \\ \pm 2m_\mp & \mp q_\pm & 0 & q_\pm \\ \mp q_\pm & 2A & q_\mp & -2D \end{pmatrix}, \tag{B.15}$$

where $m_\pm \equiv M \pm N$, $q_\pm \equiv Q \pm P$ (we have kept the original parameters M, N, etc., which are no longer the physical charges). The classification into three matrix types (strongly degenerate, weakly degenerate, non-degenerate) is invariant under this similarity transformation.

2. $f^{-1}(\infty) = 1$, $\mathrm{e}^{2\phi}(\infty) = 0$

The treatment of this case closely parallels that of case 1, with the projector α replaced by $1-\alpha$, and leads to

$$A = \eta_2' \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp 1 \\ 0 & 0 & -1 & 0 \\ 0 & \mp 1 & 0 & 0 \end{pmatrix} \tag{B.16}$$

(which is obtained from η_1' by a reflection relative to the antidiagonal and a global sign change). This is related to the matrix η by

$$\eta_2' = K_2^T \eta K_2 \,, \tag{B.17}$$

with

$$K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \mp\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & \pm\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{B.18}$$

leading to

$$B_2' = K_2^{-1} B K_2 = \begin{pmatrix} 2M & -q_\mp & 2N & \pm q_\pm \\ -q_\pm & 0 & \pm q_\pm & \mp 2d_\pm \\ -2N & \pm q_\mp & -2M & q_\pm \\ \pm q_\mp & \mp 2d_\mp & q_\mp & 0 \end{pmatrix}. \tag{B.19}$$

3. $f^{-1}(\infty) = 0$, $\mathrm{e}^{2\phi}(\infty) = 0$

In this case, $\alpha = 0$, so that the constraint (B.6) reads

$$\beta^2 = 1\,. \tag{B.20}$$

Both f^{-1} and $\mathrm{e}^{2\phi}$ go to zero as r^{-1}, so that one can choose a gauge such that asymptotically

$$\mathrm{e}^{-2\phi} \simeq f^{-1}\,, \tag{B.21}$$

and $v(\infty) = 0$, leading to

$$\beta = \begin{pmatrix} -f^{-1}\chi & f^{-1}u \\ f^{-1}u & -f^{-1}\kappa \end{pmatrix}(\infty)\,. \tag{B.22}$$

The constraint (B.20) then leads to

$$\chi^2 + u^2 \simeq \kappa^2 + u^2 \simeq f^2\,, \quad f^{-2}u(\chi+\kappa) \simeq 0\,, \tag{B.23}$$

for $r \to \infty$. These are satisfied if either

$$\chi \simeq \kappa \simeq \mp f\,, \quad u \simeq 0\,, \tag{B.24}$$

or

$$\chi \simeq -\kappa \simeq -f\cos v\,, \quad u \simeq f\sin v\,, \tag{B.25}$$

with v a real constant. These two possibilities lead, up to a gauge transformation, to $\gamma = 0$, so that

$$A = \eta'_{3\mathrm{a}} \equiv \begin{pmatrix} 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \end{pmatrix}\,, \tag{B.26}$$

in the case (B.24), or

$$A = \eta'_{3\mathrm{b}} \equiv \begin{pmatrix} 0 & 0 & \cos v & \sin v \\ 0 & 0 & \sin v & -\cos v \\ \cos v & \sin v & 0 & 0 \\ \sin v & -\cos v & 0 & 0 \end{pmatrix}\,, \tag{B.27}$$

in the case (B.25).

In the first case, the matrix transforming η into η'_{3a} is

$$K_{3a} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \pm 1 & 0 \\ 0 & 1 & 0 & \pm 1 \\ \mp 1 & 0 & 1 & 0 \\ 0 & \mp 1 & 0 & 1 \end{pmatrix}, \tag{B.28}$$

leading to

$$B'_{3a} = \pm \begin{pmatrix} 0 & 0 & 2m_\pm & -\sqrt{2}q_\mp \\ 0 & 0 & -\sqrt{2}q_\mp & 2d_\mp \\ 2m_\mp & -\sqrt{2}q_\pm & 0 & 0 \\ -\sqrt{2}q_\pm & 2d_\pm & 0 & 0 \end{pmatrix}. \tag{B.29}$$

In the second case, the transformation matrix is

$$K_{3b} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \cos\nu & \sin\nu \\ 0 & 1 & \sin\nu & -\cos\nu \\ -\cos\nu & -\sin\nu & 1 & 0 \\ -\sin\nu & \cos\nu & 0 & 1 \end{pmatrix}. \tag{B.30}$$

Simple expressions can be obtained for the corresponding matrix B in the two limiting cases $\sin\nu = 0$ ($\cos\nu = \pm 1$) and $\cos\nu = 0$ ($\sin\nu = \pm 1$).

a) $\cos\nu = \pm 1$:

$$B'_{3b} = \begin{pmatrix} 0 & -\sqrt{2}q_\mp & \pm 2m_\pm & 0 \\ -\sqrt{2}q_\pm & 0 & 0 & \mp 2d_\pm \\ \pm 2m_\mp & 0 & 0 & \sqrt{2}q_\pm \\ 0 & \mp 2d_\mp & \sqrt{2}q_\mp & 0 \end{pmatrix}. \tag{B.31}$$

b) $\sin\nu = \pm 1$:

$$B'_{3b} = \begin{pmatrix} -D+M & \mp(A+N) & -A+N\mp\sqrt{2}Q & \sqrt{2}P\pm(D+M) \\ \pm(A+N) & D-M & \sqrt{2}P\pm(D+M) & -A+N\mp\sqrt{2}Q \\ A-N\mp\sqrt{2}Q & -\sqrt{2}P\pm(D+M) & D-M & \mp(A+N) \\ -\sqrt{2}P\pm(D+M) & A-N\mp\sqrt{2}Q & \pm(A+N) & -D+M \end{pmatrix}. \tag{B.32}$$

Appendix C: Multiple null generators

Let us first show that the double commutation relations (8.2), together with the Ricci-flatness conditions (8.4) imply the commutation relations (8.5). Consider the relations (8.2) with $a = 1$, $b = 2$ and $c = 1$ or 2,

$$\big[[B, B'], B\big] = 0\,, \quad \big[[B, B'], B'\big] = 0\,, \tag{C.1}$$

with $B = B_1$, $B' = B_2$. These equations may be rewritten as

$$\boldsymbol{\mu} \wedge (\boldsymbol{\mu} \wedge \boldsymbol{\nu'}) + \boldsymbol{\mu'} \wedge (\boldsymbol{\nu} \wedge \boldsymbol{\mu}) + \boldsymbol{\nu} \wedge (\boldsymbol{\mu'} \wedge \boldsymbol{\mu}) = 0\,, \tag{C.2}$$

$$\boldsymbol{\nu'} \wedge (\boldsymbol{\nu} \wedge \boldsymbol{\mu}) + \boldsymbol{\nu} \wedge (\boldsymbol{\mu'} \wedge \boldsymbol{\nu}) + \boldsymbol{\mu} \wedge (\boldsymbol{\nu} \wedge \boldsymbol{\nu'}) = 0\,, \tag{C.3}$$

$$\boldsymbol{\mu} \wedge (\boldsymbol{\mu'} \wedge \boldsymbol{\nu'}) + \boldsymbol{\mu'} \wedge (\boldsymbol{\nu} \wedge \boldsymbol{\mu'}) + \boldsymbol{\nu'} \wedge (\boldsymbol{\mu'} \wedge \boldsymbol{\mu}) = 0\,, \tag{C.4}$$

$$\boldsymbol{\nu'} \wedge (\boldsymbol{\nu'} \wedge \boldsymbol{\mu}) + \boldsymbol{\nu} \wedge (\boldsymbol{\mu'} \wedge \boldsymbol{\nu'}) + \boldsymbol{\mu'} \wedge (\boldsymbol{\nu} \wedge \boldsymbol{\nu'}) = 0\,. \tag{C.5}$$

We also have the relations (8.4) for $a, b = 1, 2$,

$$\boldsymbol{\mu}^2 = \boldsymbol{\nu}^2\,, \quad \boldsymbol{\mu'}^2 = \boldsymbol{\nu'}^2\,, \tag{C.6}$$

$$\boldsymbol{\mu} \cdot \boldsymbol{\mu'} = \boldsymbol{\nu} \cdot \boldsymbol{\nu'}\,. \tag{C.7}$$

Using (C.6) and (C.7), (C.2) may be rewritten as

$$(\boldsymbol{\nu}^2)\boldsymbol{\nu'} - (\boldsymbol{\mu} \cdot \boldsymbol{\nu'})\boldsymbol{\mu} + (\boldsymbol{\mu'} \cdot \boldsymbol{\nu})\boldsymbol{\mu} - (\boldsymbol{\nu} \cdot \boldsymbol{\nu'})\boldsymbol{\nu} + \boldsymbol{\nu} \wedge (\boldsymbol{\mu'} \wedge \boldsymbol{\mu}) = 0\,, \tag{C.8}$$

or

$$a\boldsymbol{\mu} = \boldsymbol{b} \wedge \boldsymbol{\nu}\,, \tag{C.9}$$

with

$$a \equiv \boldsymbol{\mu}' \cdot \boldsymbol{\nu} - \boldsymbol{\mu} \cdot \boldsymbol{\nu}'\,, \quad \boldsymbol{b} \equiv \boldsymbol{\mu} \wedge \boldsymbol{\mu}' - \boldsymbol{\nu} \wedge \boldsymbol{\nu}'\,. \tag{C.10}$$

Reasoning similarly with eqs. (C.3)-(C.5), we arrive at the system

$$a\boldsymbol{\mu} = \boldsymbol{b} \wedge \boldsymbol{\nu}\,, \quad a\boldsymbol{\nu} = \boldsymbol{b} \wedge \boldsymbol{\mu}\,, \quad a\boldsymbol{\mu}' = \boldsymbol{b} \wedge \boldsymbol{\nu}'\,, \quad a\boldsymbol{\nu}' = \boldsymbol{b} \wedge \boldsymbol{\mu}'\,. \tag{C.11}$$

A consequence of (C.11), obtained by iteration, is

$$a^2\boldsymbol{\mu} = \boldsymbol{b} \wedge (\boldsymbol{b} \wedge \boldsymbol{\mu}) = (\boldsymbol{b}^2)\boldsymbol{\mu}\,, \tag{C.12}$$

and a similar equation with $\boldsymbol{\mu}$ replaced by $\boldsymbol{\nu}$. Excluding the trivial solution $\boldsymbol{\mu} = \boldsymbol{\nu} = 0$ ($B = 0$), these relations give

$$a^2 = \boldsymbol{b}^2\,. \tag{C.13}$$

We first assume $a \neq 0$, implying $\boldsymbol{b} \neq 0$. Taking the wedge product of the first eq. (C.11) by $\boldsymbol{\nu}$, and of the second eq. by $\boldsymbol{\mu}$, and taking into account $\boldsymbol{\mu} \cdot \boldsymbol{b} = \boldsymbol{\nu} \cdot \boldsymbol{b} = 0$ (which also follows from (C.11)), we obtain

$$a\boldsymbol{\mu} \wedge \boldsymbol{\nu} = -(\boldsymbol{\mu}^2)\boldsymbol{b} = (\boldsymbol{\nu}^2)\boldsymbol{b}\,, \tag{C.14}$$

together with a similar primed equation. Because of (C.6), these imply

$$\boldsymbol{\mu}^2 = \boldsymbol{\nu}^2 = 0\,, \quad \boldsymbol{\nu} = c\boldsymbol{\mu}\,, \quad \boldsymbol{\nu}' = c'\boldsymbol{\mu}'\,. \tag{C.15}$$

Inserting these two last equations into (C.11), one obtains

$$a\boldsymbol{\mu} = c\boldsymbol{b}\wedge\boldsymbol{\nu} = c^{-1}\boldsymbol{b}\wedge\boldsymbol{\nu}\,,$$

and a similar primed equation, so that $c^2 = c'^2 = 1$. Then from (C.10) $a = (c-c')\boldsymbol{\mu}\cdot\boldsymbol{\mu}' = c(1-cc')\boldsymbol{\mu}\cdot\boldsymbol{\mu}'$, which vanishes from (C.7), contrary to our hypothesis.

It follows that $a = 0$. Then necessarily also $\boldsymbol{b} = 0$ (if one assumes $\boldsymbol{b} \neq 0$, then from (C.11) the four vectors $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, $\boldsymbol{\mu}'$, $\boldsymbol{\nu}'$ are all collinear with the same vector $\boldsymbol{b}$, leading to $\boldsymbol{b} = 0$). The four equations $a = 0$, $\boldsymbol{b} = 0$,

$$\boldsymbol{\mu}'\cdot\boldsymbol{\nu} = \boldsymbol{\mu}\cdot\boldsymbol{\nu}'\,, \quad \boldsymbol{\mu}\wedge\boldsymbol{\mu}' = \boldsymbol{\nu}\wedge\boldsymbol{\nu}' \tag{C.16}$$

are equivalent to $[B,B'] = 0$.

Now we discuss the system of equations (C.6), (C.7), (C.16) for a two-potential BPS solution. Putting $\boldsymbol{\mu}_\pm = \boldsymbol{\mu}\pm\boldsymbol{\nu}$, $\boldsymbol{\mu}'_\pm = \boldsymbol{\mu}'\pm\boldsymbol{\nu}'$, this system may be rewritten as

$$\boldsymbol{\mu}_+\cdot\boldsymbol{\mu}_- = \boldsymbol{\mu}_+\cdot\boldsymbol{\mu}'_- = \boldsymbol{\mu}'_+\cdot\boldsymbol{\mu}_- = \boldsymbol{\mu}'_+\cdot\boldsymbol{\mu}'_- = 0 \tag{C.17}$$

$$\boldsymbol{\mu}_+\wedge\boldsymbol{\mu}'_- + \boldsymbol{\mu}_-\wedge\boldsymbol{\mu}'_+ = 0\,. \tag{C.18}$$

Taking successively the wedge product of the vector equation (C.18) with the four vectors $\boldsymbol{\mu}_+$, $\boldsymbol{\mu}_-$, $\boldsymbol{\mu}'_+$, $\boldsymbol{\mu}'_-$, we obtain the secondary system

$$\begin{aligned} &(\boldsymbol{\mu}_+^2)\boldsymbol{\mu}'_- - (\boldsymbol{\mu}_+\cdot\boldsymbol{\mu}'_+)\boldsymbol{\mu}_- = 0\,, \quad (\boldsymbol{\mu}_-^2)\boldsymbol{\mu}'_+ - (\boldsymbol{\mu}_-\cdot\boldsymbol{\mu}'_-)\boldsymbol{\mu}_+ = 0\,, \\ &(\boldsymbol{\mu}'^2_+)\boldsymbol{\mu}_- - (\boldsymbol{\mu}'_+\cdot\boldsymbol{\mu}_+)\boldsymbol{\mu}'_- = 0\,, \quad (\boldsymbol{\mu}'^2_-)\boldsymbol{\mu}_+ - (\boldsymbol{\mu}'_-\cdot\boldsymbol{\mu}_-)\boldsymbol{\mu}'_+ = 0\,. \end{aligned} \tag{C.19}$$

First assume that none of the vectors $\boldsymbol{\mu}_\pm$, $\boldsymbol{\mu}'_\pm$ vanishes. If also all these vectors are non-null, then from the system (C.19) $\boldsymbol{\mu}'_+$ is proportional to $\boldsymbol{\mu}_+$ and $\boldsymbol{\mu}'_-$ is proportional to $\boldsymbol{\mu}_-$. In that case, by replacing the original harmonic potentials τ and τ' by suitable linear combinations of τ and τ', one can translate $\boldsymbol{\mu}'_+$ or $\boldsymbol{\mu}'_-$ to zero, contrary to our assumption. If one of the vectors, e.g. $\boldsymbol{\mu}_+$ is null, $\boldsymbol{\mu}_+^2 = 0$, then the first and third equations (C.19) give also $\boldsymbol{\mu}_+\cdot\boldsymbol{\mu}'_+ = \boldsymbol{\mu}'^2_+ = 0$, so that $\boldsymbol{\mu}'_+$ must be proportional to $\boldsymbol{\mu}_+$, and again may be translated to zero by a redefinition of the harmonic potentials τ and τ'. So we conclude that at least one of the four vectors $\boldsymbol{\mu}_\pm$, $\boldsymbol{\mu}'_\pm$ must vanish (up to a redefinition of the harmonic potentials).

Assume that e.g. $\boldsymbol{\mu}'_+ = 0$. Then, from Eq. (C.18) $\boldsymbol{\mu}_+ = c\boldsymbol{\mu}'_-$. There are two possibilities:

a) $\underline{c=0}$. Then

$$\boldsymbol{\mu}'_+ = \boldsymbol{\mu}_+ = 0\,, \tag{C.20}$$

which solves all the equations (C.17) and (C.18). Both matrices B and B' belong to the strongly degenerate subcase 1.

b) $\underline{c \neq 0}$. Then from (C.17) one obtains $\boldsymbol{\mu}_+ \cdot \boldsymbol{\mu}_- = 0$ and $\boldsymbol{\mu}_+^2 = 0$, so that the matrix B is degenerate and B' belong to the lightlike sector of the strongly degenerate subcase 1. This can be further divided into three subcases. In the first ($\boldsymbol{\mu}_- = 0$), B is also in the lightlike sector of the strongly degenerate subcase 1, with

$$\boldsymbol{\mu}_+^2 = 0\,, \quad \boldsymbol{\mu}_- = \boldsymbol{\mu}'_+ = 0\,, \quad \boldsymbol{\mu}'_- = b\boldsymbol{\mu}_+ \tag{C.21}$$

($b = c^{-1}$).

In the second subcase, $\boldsymbol{\mu}_- \propto \boldsymbol{\mu}_+$, B is strongly degenerate subcase 2,

$$\boldsymbol{\mu}_+^2 = 0\,, \quad \boldsymbol{\mu}_- = a\boldsymbol{\mu}_+\,, \quad \boldsymbol{\mu}'_+ = 0\,, \quad \boldsymbol{\mu}'_- = b\boldsymbol{\mu}_+\,. \tag{C.22}$$

However, this second subcase is equivalent to the first, as can be shown by taking the linear combinations $\tilde{B} = B - (a/b)B'$, $\tilde{B}' = B'$, leading to $\tilde{\mu}_+ = \mu_+$ (so that $\tilde{\mu}_+^2 = 0$), and $\tilde{\mu}_- = 0$.

In the third subcase, B is weakly degenerate,

$$\boldsymbol{\mu}_+^2 = 0\,, \quad \boldsymbol{\mu}_+ \cdot \boldsymbol{\mu}_- = 0\,, \quad \boldsymbol{\mu}'_+ = 0\,, \quad \boldsymbol{\mu}'_- = b\boldsymbol{\mu}_+\,. \tag{C.23}$$

Note that in the case of a weakly degenerate matrix B, $\boldsymbol{\nu} \wedge \boldsymbol{\lambda} = -(\boldsymbol{\mu}^2)\boldsymbol{\mu}_+$, $\boldsymbol{\mu} \wedge \boldsymbol{\lambda} = (\boldsymbol{\mu}^2)\boldsymbol{\mu}_+$, with $\boldsymbol{\mu}^2 \neq 0$, so that from (5.33) one can identify

$$B' = -\frac{b}{4\boldsymbol{\mu}^2} B^3\,. \tag{C.24}$$

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